Journal of Basics and Applied Sciences Research (JOBASR)
ISSN (print): 3026-9091, ISSN (online): 1597-9962
Volume 3(6) November 2025
DOI: https://dx.doi.org/10.4314/jobasr.v3i6.29

Modification of Logarithmic-Type Estimators of Population Variance in Two-Phase-Successive Sampling with Random Non-Response and Measurement Errors

Musa Salihu[1*], Abubakar Danbaba[2], Ahmed Audu[3] & Abdulkarim Bello[4]
[1]National Bureau of Statistics, Nigeria
[2,3]Department of Statistics, Usmanu Danfodiyo University, Sokoto, Nigeria
[4]Department of Computer Science Usmanu Danfodiyo University, Sokoto, Nigeria
[*]Corresponding Author Email: musakudaru1@gmail.com

ABSTRACT
The estimation of population variance is a critical undertaking in survey sampling, with significant implications across fields such as agriculture, economics, and public health. However, the precision of such estimates is often compromised by the dual challenges of non-response and measurement errors, particularly in successive sampling designs. Furthermore, many existing efficient estimators rely on the unrealistic and costly assumption of known population parameters for auxiliary variables. This study therefore proposes modified variance estimators designed to overcome these limitations. The research adopted a two-phase sampling approach within a successive sampling framework to develop logarithmic-type estimators that do not require known population parameters for the auxiliary variables. The biases and Mean Square Errors (MSEs) of the proposed estimators were derived theoretically to first-order approximation. A comprehensive simulation study was conducted to evaluate their performance using Absolute Relative Bias (ARB), MSE, and Percentage Relative Efficiency (PRE) as performance metrics, comparing them against a conventional estimator. The results demonstrate that the proposed estimators, particularly those utilizing estimates from the large first-phase sample, are highly efficient. PRE values consistently exceeded 200% in many scenarios, indicating a dramatic improvement in precision over the conventional method, while maintaining low bias. The study concludes that the modified estimators provide a viable, cost-effective, and accessible framework for estimating population variance with enhanced precision in the presence of non-response and measurement errors, thereby offering a significant contribution to survey sampling theory and practice.

Keywords:
Two-phase Sampling; Successive Sampling; Random Nonresponse; Mean square error; Measurement Error.

INTRODUCTION

Sampling theory is a fundamental concept in statistics that enables researchers to make valid inferences about a population based on a representative subset of its units (Cochran, 1977). In many real-life situations, it is impractical or impossible to examine every unit in a population; hence, sampling provides a practical solution by allowing conclusions to be drawn from selected samples. The use of auxiliary information variables that are strongly correlated with the study variable plays a vital role in improving the precision and efficiency of statistical estimators.

Among population parameters, the estimation of variance is particularly important, as it quantifies the degree of variability or dispersion within a population. Accurate estimation of variance is essential in diverse fields such as agriculture, biology, business, health sciences, and manufacturing, where understanding variability informs better decision-making and policy formulation.

Over the years, several scholars have contributed significantly to the development of improved estimators for population variance using auxiliary information. Studies by Garcia and Cebrian (1996), Upadhyaya et al. (2004), Chandra and Singh (2005),

How to cite this article: Musa S., Abubakar D., Ahmed A. & Abdulkarim B. (2025). Modification of Logarithmic-Type Estimators of Population Variance in Two-Phase-Successive Sampling with Random Non-Response and Measurement Errors. *Journal of Basics and Applied Sciences Research*, *3*(6), 292-319. https://dx.doi.org/10.4314/jobasr.v3i6.29

Kadilar and Cingi (2007), Sharma and Singh (2013), Yadav and Kadilar (2014), Ahmed et al. (2016a,b,c) and Ahmad et al. (2023), among others, have proposed ratio, product, regression, and generalized estimators to enhance efficiency and minimize mean squared error (MSE). These approaches have been applied across various sampling frameworks to obtain more reliable variance estimates. However, most of these estimators were derived under simple random sampling assumptions, presuming complete response from all sampled units an assumption that often fails in practical survey contexts due to non-sampling errors such as non-response and measurement error.

In modern survey practice, non-response has emerged as a significant challenge, particularly in successive sampling, where data are collected from the same population over multiple occasions. Successive (or repeated) sampling first introduced by Jessen (1942) and later expanded by Yates (1949), Patterson (1950), and Eckler (1955) involves partial replacement of sampling units between survey rounds, reducing costs and improving continuity in longitudinal data collection. This technique has wide applications in socio-economic and scientific studies that monitor changes in populations over time. However, the presence of non-response and measurement errors complicates variance estimation, as traditional estimators that assume full response become biased and inefficient. Recent contributions by Audu *et al.* (2025) addressed these challenges by developing logarithmic-type estimators for population variance in successive sampling, capable of accommodating non-response and measurement errors. Their estimators utilized information from both current and previous surveys to improve efficiency. Nonetheless, their approach assumed that the population mean and variance of the auxiliary variable were known an assumption that may not hold in practice when such information is incomplete or costly to obtain. To bridge this gap, the present study extends the work of Audu *et al.* (2025) by proposing improved estimators for population variance under two-phase sampling, where estimates of auxiliary parameters are derived from a large preliminary sample. This approach enhances practical applicability by addressing the unavailability of complete auxiliary information, thereby contributing to more efficient and reliable estimation of population variance in successive sampling designs.

The paper is organized as follows: Section 1 discussed the background and significance of the study, as well as the importance of the proposed methods. Section 2 introduces basic notations, sample structures, and random non-response models/probability functions. Section 3 presents the proposed logarithmic-type estimators along with their theoretical properties. Section 4 compares the performance of the proposed estimators against the estimator of population variance obtained through a linear combination of sample variance and ratio estimator numerically through simulation studies. Finally, Section 5 provides the conclusion and offers some recommendations based on the findings.

Notations and Sample Structure

Consider a finite population $U=\{U_1,U_2,....U_N\}$ of size N sampled over two occasions. Let the study variable be denoted by X and Y on the first and second occasions, respectively. We assume information on an auxiliary variable Z is available on both occasions, and its population mean is known. We further assume that response occurs on both occasions.

On the first occasion, a preliminary simple random sampling without replacement (SRSWOR) sample of size n is drawn from the population, where r_1 units do not respond. From the responding part of this sample, a second-stage SRSWOR sample of size $m=n\lambda$ is drawn, where r_3 units do not respond. This sample is matched or retained for the second occasion, and information on the study variable Y is collected. Additionally, a fresh sample of size $u=n-m=n\mu$ is drawn from the population using SRSWOR, and information on Y is collected again, where r_3 units do not respond. Let λ and $\mu(\lambda+\mu)=1$ be the fractions of matched and fresh samples, respectively, on the second occasion (Audu et al.; 2025).

Henceforth, the following notations have been used:

$\bar{Y}$: Mean of the population for the study variable Y.

$\bar{X},\bar{Z}$: Means of the population for the auxiliary variables X and Z respectively.

S_y^2 : The population variances of the study variable Y respectively.

S_x^2,S_z^2 : The population variances of the auxiliary variables X and Z respectively.

$\bar{y}_m,\bar{y}_u,\bar{x}_n,\bar{x}_m,\bar{z}_n,\bar{z}_m,\bar{z}_u$: Means of the sample for variables Y, X and Z respectively

$\bar{z}_n=n^{-1}\sum_{i=1}^{n}z_i$, $\bar{z}_m=m^{-1}\sum_{i=1}^{m}z_i$,

$\bar{z}_u=u^{-1}\sum_{i=1}^{u}z_i$: Means of the sample for auxiliary variable Z based on samples of size n, m and u respectively.

$s_{z_n}^2=(n-1)^{-1}\sum_{i=1}^{n}(z_i-\bar{z}_n)^2$,

$s_{z_m}^2=(m-1)^{-1}\sum_{i=1}^{m}(z_i-\bar{z}_m)^2$,

$s_{z_u}^2=(u-1)^{-1}\sum_{i=1}^{u}(z_i-\bar{z}_u)^2$: The sample variances of auxiliary variable Z based on samples of size n, m and u respectively.

$\bar{y}_{m-\eta_2} = (m-\eta_2)^{-1} \sum_{i=1}^{m-\eta_2} y_i$,

$\bar{y}_{u-\eta_3} = (u-\eta_3)^{-1} \sum_{i=1}^{u-\eta_3} y_i$: The sample means of Y based on the number of responding units in the samples of size m and u respectively on the first and second occasion.

$s^2_{y_{m-\eta_2}} = (m-\eta_2-1)^{-1} \sum_{i=1}^{m-\eta_2} \left(y_i - \bar{y}_{m-\eta_2}\right)^2$,

$s^2_{y_{u-\eta_3}} = (u-\eta_3-1)^{-1} \sum_{i=1}^{u-\eta_3} \left(y_i - \bar{y}_{u-\eta_3}\right)^2$: The sample variances of Y based on the number of responding units in the samples of size m and u respectively on the first and second occasion.

$\bar{x}_{n-\eta_1} = (n-\eta_1)^{-1} \sum_{i=1}^{n-\eta_1} y_i$,

$\bar{x}_{m-\eta_2} = (m-\eta_2)^{-1} \sum_{i=1}^{m-\eta_2} y_i$: Means of sample for X based on the number of respondents in the samples of size n and m on the first and second occasion respectively.

$s^2_{x_{n-\eta_1}} = (n-\eta_1-1)^{-1} \sum_{i=1}^{n-\eta_1} \left(x_i - \bar{x}_{n-\eta_1}\right)^2$,

$s^2_{x_{m-\eta_2}} = (m-\eta_2-1)^{-1} \sum_{i=1}^{m-\eta_2} \left(x_i - \bar{x}_{m-\eta_2}\right)^2$: The sample variances of X based on the number of respondents in the samples of size n and m respectively on the first and second occasion.

$c_{y_{m-\eta_2}} = \frac{s_{y_{m-\eta_2}}}{\bar{y}_{m-\eta_2}}, c_{x_{m-\eta_2}} = \frac{s_{x_{m-\eta_2}}}{\bar{x}_{m-\eta_2}}, c_{x_{n-\eta_1}} = \frac{s_{x_{n-\eta_1}}}{\bar{x}_{n-\eta_1}}, c_{z_m} = \frac{s_{z_m}}{\bar{z}_m}, c_{z_n} = \frac{s_{z_n}}{\bar{z}_n}, c_{z_u} = \frac{s_{z_u}}{\bar{z}_u}$

: The sample coefficients of variation for the variables Y, X and Z respectively.

$\rho_{xy}, \rho_{yz}, \rho_{zx}$: The population correlation coefficients between the variables shown in suffice.

S^2_X, S^2_Y, S^2_Z: The population mean squares of the variables X, Y and Z respectively.

$C_X, C_Y, C_Z, C_{XY}, C_{YZ}, C_{ZX}$: Coefficient of variations for X, Y and Z.

Non-response Probability Model

Consider $\eta_1 \{\eta_1 = 0,1,2,....,n-2\}$ as the number of non-respondents in the sample S_n of size n due to random non-response. Also, consider $\eta_2 \{\eta_2 = 0,1,2,....,m-2\}$ as the number of non-respondents in the sample S_m of size m. Finally, let $\eta_3 \{\eta_3 = 0,1,2,....,u-2\}$ be the number of non-respondents in the sample of size u. It is assumed that $0 \le \eta_1 \le (n-2)$, $0 \le \eta_2 \le (m-2)$, and $0 \le \eta_3 \le (u-2)$. If π_1, π_2, π_3 are the probabilities of non-response among the $(n-2)$, $(m-2)$ and $(u-2)$ respectively, then η_1, η_1 and η_3 have the following probability mass function respectively as in (1), (2) and (3) respectively (Audu et al.; 2025).

$$p(\eta_1) = \frac{n-\eta_1}{n(1-\pi_1)+2\pi_1} C^{n-2}_{\eta_1} \pi_1^{\eta_1} \left(1-\pi_1^{n-\eta_1-2}\right);$$

$$\eta_1 = 0,1,2,........,n-2 \qquad (1)$$

$$p(\eta_2) = \frac{m-\eta_2}{m(1-\pi_2)+2\pi_2} C^{m-2}_{\eta_2} \pi_2^{\eta_2} \left(1-\pi_2^{m-\eta_2-2}\right);$$

$$\eta_2 = 0,1,2,........,m-2 \qquad (2)$$

$$p(\eta_3) = \frac{u-\eta_3}{u(1-\pi_3)+2\pi_3} C^{u-2}_{\eta_3} \pi_3^{\eta_3} \left(1-\pi_3^{u-\eta_3-2}\right);$$

$$\eta_2 = 0,1,2,........,m-2 \qquad (3)$$

where $C^{n-2}_{\eta_1}$, $C^{m-2}_{\eta_2}$ and $C^{u-2}_{\eta_3}$ are the total possible ways of getting $\eta_i (i=1,2,3)$ out of $(n-2)$, $(m-2)$ and $(u-2)$ respectively.

MATERIALS AND METHODS

Proposed Estimators

Inspired by the preceding discussions and the estimation methods utilized by Audu et al. (2024), the following logarithmic-type estimators of population variance denoted by Ψ_1, Ψ_2 based on the sample S_m of size m common to both the occasions are suggested as in (4) and (5) respectively.

$$\Psi^*_1 = s^{*2}_{y_m} + \log\left(1 + k^*_1\left(\bar{x}^*_n - \bar{x}^*_m\right)\right) \qquad (4)$$

$$\Psi^*_2 = s^{*2}_{y_m} + \log\left(1 + k^*_2\left(s^{*2}_{x_n} - s^{*2}_{x_m}\right)\right) \qquad (5)$$

where $k_j, j=1,2$ are suitable scalar quantities to be determined by minimizing the mean square error of the estimators $t_j, j=1,2$,

$$\bar{x}^*_m = \bar{x}_{m-\eta_2} + \log\left(1 + \frac{\bar{z}^* - \bar{z}_m}{\bar{z}^* + \bar{z}_m}\right),$$

$$s_{y_m}^{*2}=s_{y_{m-\eta_2}}^{2}+\log\left(1+\frac{s_z^{2*}-s_{z_m}^{2}}{s_z^{2*}+s_{z_m}^{2}}\right)$$

$$s_{x_m}^{*2}=s_{x_{m-\eta_2}}^{2}+\log\left(1+\frac{s_z^{2*}-s_{z_m}^{2}}{s_z^{2*}+s_{z_m}^{2}}\right), \quad , \quad ,$$

$$\bar{x}_n^{*}=\bar{x}_{n-\eta_1}+\log\left(1+\frac{\bar{z}^{*}-\bar{z}_n}{\bar{z}^{*}+\bar{z}_n}\right),$$

$$s_{x_n}^{*2}=s_{x_{n-\eta_1}}^{2}+\log\left(1+\frac{s_z^{2*}-s_{z_n}^{2}}{s_z^{2*}+s_{z_n}^{2}}\right).$$

Similarly, we suggest the following logarithmic-type estimators θ_1,θ_2 of population variance based on the fresh sample S_u of size u drawn on the current occasion as in (6) and (7)

$$\theta_1^{*}=s_{y_{u-\eta_3}}^{2}+\log\left(1+a_1\frac{\bar{z}^{*}-\bar{z}_u}{\bar{z}^{*}+\bar{z}_u}\right) \tag{6}$$

$$\theta_2^{*}=s_{y_{u-\eta_3}}^{2}+\log\left(1+a_2\frac{s_z^{2*}-s_{z_u}^{2}}{s_z^{2*}+s_{z_u}^{2}}\right) \tag{7}$$

where $a_i, i=1,2$ are suitable scalar quantities to be determined by minimizing the mean square error of the estimators $\theta_j^{*}, j=1,2,$

Properties of the Proposed Estimators

To derive the bias and mean square errors of proposed estimators up to the first order of aproximations under large sample assumptions, the following transformations are defined. $\bar{z}^{*}=\bar{Z}(1+e_0)$, $\bar{x}_{m-\eta_2}=\bar{X}(1+e_1)$, $\bar{x}_{n-\eta_1}=\bar{X}(1+e_2)$, $s_z^{*2}=S_z^2(1+e_3)$, $\bar{z}_m=\bar{Z}(1+e_4)$, $\bar{z}_n=\bar{Z}(1+e_5)$, $\bar{z}_u=\bar{Z}(1+e_6)$, $s_{y_{m-\eta_2}}^2=S_y^2(1+e_7)$, $s_{z_m}^2=S_z^2(1+e_8)$, $s_{z_n}^2=S_z^2(1+e_9)$, $s_{x_{n-\eta_1}}^2=S_x^2(1+e_{10})$, $s_{y_{u-\eta_3}}^2=S_y^2(1+e_{11})$, $s_{z_u}^2=S_z^2(1+e_{12})$, $s_{x_{m-\eta_2}}^2=S_x^2(1+e_{13})$.

Such that $E(e_i)=0$ and $|e_i|\le 1,\ \forall\ i=0,1,2,......,13$, then, the following expectations hold.

$E(e_0^2)=\kappa_7C_z^2$, $E(e_1^2)=\kappa_2C_x^2$, $E(e_2^2)=\kappa_1C_x^2$,
$E(e_3^2)=\kappa_7(\lambda_{004}-1)$, $E(e_4^2)=\kappa_5C_z^2$,
$E(e_5^2)=\kappa_4C_z^2$, $E(e_6^2)=\kappa_6C_z^2$,
$E(e_7^2)=\kappa_2(\lambda_{400}-1)$, $E(e_8^2)=\kappa_5(\lambda_{004}-1)$,
$E(e_9^2)=\kappa_4(\lambda_{004}-1)$, $E(e_{10}^2)=\kappa_1(\lambda_{040}-1)$,
$E(e_{11}^2)=\kappa_3(\lambda_{400}-1)$, $E(e_{12}^2)=\kappa_6(\lambda_{004}-1)$,
$E(e_{13}^2)=\kappa_2(\lambda_{040}-1)$,
$E(e_0e_1)=\kappa_7C_{xz}$, $E(e_0e_2)=\kappa_7C_{xz}$,
$E(e_0e_4)=\kappa_7C_z^2$, $E(e_0e_5)=\kappa_7C_z^2$,
$E(e_0e_7)=\kappa_7C_z\lambda_{201}$, $E(e_0e_8)=\kappa_7C_z\lambda_{003}$,
$E(e_0e_9)=\kappa_4C_z\lambda_{003}$, $E(e_0e_{10})=\kappa_7C_z\lambda_{003}$,
$E(e_0e_{13})=\kappa_7C_z\lambda_{021}$, $E(e_1e_2)=\kappa_1C_x^2$,
$E(e_1e_4)=\kappa_5C_{xz}$, $E(e_1e_5)=\kappa_4C_{xz}$,
$E(e_1e_7)=\kappa_2C_x\lambda_{210}$, $E(e_1e_8)=\kappa_5C_x\lambda_{012}$,
$E(e_1e_9)=\kappa_4C_x\lambda_{012}$, $E(e_1e_{10})=\kappa_1C_x\lambda_{030}$,
$E(e_1e_{13})=\kappa_2C_x\lambda_{030}$, $E(e_2e_4)=\kappa_1C_{xz}$,
$E(e_2e_5)=\kappa_4C_{xz}$, $E(e_2e_7)=\kappa_1C_x\lambda_{210}$,
$E(e_2e_8)=\kappa_1C_x\lambda_{012}$, $E(e_2e_9)=\kappa_4C_x\lambda_{012}$,
$E(e_2e_{10})=\kappa_1C_x\lambda_{030}$ $E(e_2e_{13})=\kappa_1C_x\lambda_{030}$,
$E(e_3e_6)=\kappa_7C_z\lambda_{003}$, $E(e_3e_{11})=\kappa_7(\lambda_{202}-1)$,
$E(e_3e_{12})=\kappa_7(\lambda_{004}-1)$, $E(e_4e_5)=\kappa_4C_z^2$,
$E(e_4e_7)=\kappa_5C_z\lambda_{201}$, $E(e_4e_8)=\kappa_5C_z\lambda_{003}$,
$E(e_4e_9)=\kappa_4C_z\lambda_{003}$, $E(e_4e_{10})=\kappa_1C_z\lambda_{021}$,
$E(e_4e_{13})=\kappa_4C_z\lambda_{021}$, $E(e_5e_7)=\kappa_4C_z\lambda_{201}$,
$E(e_5e_8)=\kappa_4C_z\lambda_{003}$, $E(e_5e_9)=\kappa_4C_z\lambda_{003}$,
$E(e_5e_{10})=\kappa_4C_z\lambda_{021}$, $E(e_5e_{13})=\kappa_4C_z\lambda_{021}$,
$E(e_6e_{11})=\kappa_6C_z\lambda_{201}$, $E(e_6e_{12})=\kappa_6C_z\lambda_{003}$,
$E(e_7e_8)=\kappa_5(\lambda_{202}-1)$, $E(e_7e_9)=\kappa_4(\lambda_{202}-1)$,
$E(e_7e_{10})=\kappa_1(\lambda_{220}-1)$, $E(e_7e_{13})=\kappa_2(\lambda_{220}-1)$,

$E(e_8e_9)=\kappa_4(\lambda_{004}-1)$, $E(e_8e_{10})=\kappa_2(\lambda_{022}-1)$, $E(e_8e_{13})=\kappa_5(\lambda_{022}-1)$, $E(e_9e_{10})=\kappa_4(\lambda_{022}-1)$, $E(e_9e_{13})=\kappa_4(\lambda_{022}-1)$, $E(e_{10}e_{13})=\kappa_1(\lambda_{040}-1)$, $E(e_{11}e_{12})=\kappa_6(\lambda_{202}-1)$.

where $\kappa_1=\dfrac{1}{nq_1+2p_1}-\dfrac{1}{N}$, $\kappa_2=\dfrac{1}{mq_2+2p_2}-\dfrac{1}{N}$, $\kappa_3=\dfrac{1}{uq_3+2p_3}-\dfrac{1}{N}$, $\kappa_4=\dfrac{1}{n}-\dfrac{1}{N}$, $\kappa_5=\dfrac{1}{m}-\dfrac{1}{N}$, $\kappa_6=\dfrac{1}{u}-\dfrac{1}{N}$, $\kappa_7=\dfrac{1}{n_1}-\dfrac{1}{N}$,

$$\lambda_{pqs}=\frac{1}{N-1}\sum_{i=1}^{N}(y_i-\bar{Y})^p(x_i-\bar{X})^q(z_i-\bar{Z})^s$$

3.1.1 *Biases and MSEs of Estimator* $\Psi_i^*,\ i=1,2$.

Using transformations in subsection 3.1, estimators $\Psi_i^*,\ i=1,2$ in (4) and (5) can be expressed as in (8) and (9) respectively,

$$\begin{aligned}\Psi_1^*-S_y^2 &= S_y^2e_7+\frac{1}{2}e_3-\frac{e_8}{2}-\frac{3}{8}e_3^2+\frac{1}{8}e_8^2+\frac{1}{8}e_3e_8\\ &+k_1^*\left(\frac{1}{2}e_3-\frac{1}{2}e_8\,\frac{1}{2}e_0+\frac{1}{2}e_4+\bar{X}(e_2-e_1)-\frac{1}{8}\left(3e_3^2-e_8^2-e_0+e_4\right)+\frac{1}{4}\left(e_3e_8-e_0e_4\right)\right)-\\ &k_1^{*2}\left(\bar{X}^2\frac{1}{4}\left(e_2^2+e_1^2-2e_1e_2\right)+\bar{X}\begin{pmatrix}e_2e_3-e_1e_3-e_2e_8+e_1e_8+\\ e_0e_2+e_0e_1+e_2e_4+e_1e_4\end{pmatrix}+\frac{1}{4}\left(e_3^2+e_8^2+e_0^2+e_4^2\right)+\frac{1}{2}\left(e_3e_8-e_0e_3+e_3e_4+e_0e_8-e_4e_8\right)\right)\end{aligned} \quad (8)$$

$$\begin{aligned}\Psi_2^*-S_y^2 &= S_y^2e_7+\frac{1}{2}(e_3-e_8)-\frac{1}{8}\left(e_3^2-e_8^2\right)-\frac{1}{4}e_3e_8+\\ &k_1^*\begin{pmatrix}\frac{1}{2}(e_8-e_9)+s_x^2(e_{10}-e_{13})-\frac{1}{4}(e_3e_9-e_3e_8)+\\ \frac{1}{8}\left(3e_9^2+e_8^2-e_0+e_4\right)+\frac{1}{4}(e_3e_8-e_0e_4)\end{pmatrix}\\ &-k_1^{*2}\left(s_x^2\,\frac{1}{2}\begin{pmatrix}e_{10}^2+s_x^2e_{13}^2+e_8e_{10}-e_8e_{13}-\\ e_9e_{10}+e_9e_{13}+s_x^2e_{10}e_{13}\end{pmatrix}-\frac{1}{4}e_8e_9\right)\end{aligned} \quad (9)$$

Taking expectations of (8), (9) and apply the results of expected values of the error terms in subsection 3.1, we obtained the biases of estimator $\Psi_i^*,\ i=1,2$ as in (10) and (11) respectively.

$$Bias(\Psi_1^*)=H_1+k_1^*H_2-k_1^{*2}H_3 \quad (10)$$

$$Bias(\Psi_1^*)=J_1+k_1^*J_2-k_1^{*2}J_3 \quad (11)$$

Where $H_1=\frac{1}{8}(\lambda_{004}-1)(k_5-2k_7)$,

$H_2=\frac{1}{8}(\lambda_{004}-1)(k_5-5k_7)-\frac{1}{8}\kappa_7C_z^2$,

$$H_3=\frac{1}{2}\begin{pmatrix}\bar{X}\begin{pmatrix}c_x^2(k_1-k_2)+C_x\lambda_{012}(k_1-k_5)-\\ C_{xz}(k_1-k_5)\end{pmatrix}-\\ \frac{1}{4}C_z^2(k_5-k_7)-\frac{1}{2}C_z\lambda_{003}(k_7-k_5)+\\ \frac{1}{4}(\lambda_{004}-1)(k_7+k_5)\end{pmatrix}$$

$J_1=s_y^2+\frac{1}{8}(\lambda_{004}-1)(k_5-3k_7)$, $J_2=\frac{1}{8}(\lambda_{004}-1)(3k_4+k_5-3k_7)$

$$J_3=\frac{1}{2}\begin{pmatrix}\frac{1}{4}(\lambda_{004}-1)(k_5-k_4)+s_x^2(\lambda_{040}-1)\\ (k_1+S_x^2\kappa_2)+S_x^2(\lambda_{022}-1)(k_2-k_5)\end{pmatrix}$$

Squaring both sides of (10), (11) and take expectation of the resulting equations to obtain the MSEs of estimators $\Psi_i,\ i=1,2$ as in (12) and (13) respectively.

$$MSE\left(\Psi_1^*\right)=G_1+k_1^*G_2+k_1^{*2}G_3 \quad (12)$$

$$MSE\left(\Psi_2^*\right)=L_1+k_1^*L_2+k_1^{*2}L_3 \quad (13)$$

where, $G_1 = S_y^4\kappa_2\left(\lambda_{400}-1\right)+\frac{1}{4}\left(\lambda_{004}-1\right)\left(\kappa_5-\kappa_7\right)+ S_y^2\left(\lambda_{202}-1\right)\left(\kappa_7-\kappa_5\right)$,

$$G_2 = \frac{1}{4}\left(\left(\lambda_{004}-1\right)\left(\kappa_5-\kappa_7\right)+C_z^2\left(\kappa_5-\kappa_7\right)\right) +\bar{X}^2C_x^2\left(\kappa_2-\kappa_1\right)+ \frac{1}{2}C_z\lambda_{003}\left(\kappa_7-\kappa_5\right)+ \bar{X}C_x\lambda_{012}\left(\kappa_5-\kappa_1\right)+\bar{X}C_{xz}\left(\kappa_1-\kappa_5\right),$$

$$G_3 = S_y^2\begin{pmatrix}\left(\lambda_{202}-1\right)\left(\kappa_7-\kappa_5\right)+C_z\lambda_{201}\left(\kappa_5-\kappa_7\right)\\ +2\bar{X}C_x\lambda_{201}\left(\kappa_1-\kappa_2\right)\end{pmatrix}+ \frac{1}{2}\left(\lambda_{004}-1\right)\left(\kappa_5-\kappa_7\right)+ \frac{1}{2}C_z\lambda_{003}\left(\kappa_7-\kappa_5\right)+\bar{X}C_x\lambda_{012}\left(\kappa_5-\kappa_1\right),$$

$$L_1=S_y^4\kappa_2\left(\lambda_{400}-1\right)+\frac{1}{4}\left(\lambda_{004}-1\right)\left(\kappa_7+\kappa_5\right)$$

$$L_2=\frac{1}{4}\left(\lambda_{004}-1\right)\left(\kappa_5+\kappa_4\right)-S_x^4\left(\lambda_{040}-1\right) \left(\kappa_2-2\kappa_1\right)+S_x^2\kappa_1\left(\lambda_{040}-1\right)-\frac{1}{2}\kappa_5c_x\lambda_{012} +S_x^2\left(\lambda_{022}-1\right)\left(\kappa_2-\kappa_5\right)$$

$$L_3 = S_y^2\left(\lambda_{202}-1\right)\left(\kappa_5-\kappa_4\right)+ 2S_x^2S_y^2\left(\lambda_{220}-1\right)\left(\kappa_1-\kappa_2\right)+ \frac{1}{2}\left(\lambda_{004}-1\right)\left(\kappa_4-\kappa_5\right)+ S_x^2\left(\lambda_{022}-1\right)\left(\kappa_5-\kappa_2\right)$$

Differentiating (12) and (13) with respect to k_1^*,k_2^* respectively, equate the results to zeros and solve for k_1^*,k_2^*, the expressions for the optimum values of k_1^*,k_2^* denoted by $k_{1(opt)}^*,k_{2(opt)}^*$ are obtained as in (14) and (15) respectively.

$$k_{1(opt)}^*=-G_3 2G_2^{-1} \quad (14)$$

$$k_{2(opt)}^*=-L_3L_2^{-1} \quad (15)$$

Substituting $k_{1(opt)}^*$ and $k_{2(opt)}^*$ in (12) and (13) respectively, minimum MSEs of $\Psi_i^*,\ i=1,2$ as in (16) and (17)

$$MSE\left(\Psi_1^*\right)_{min}=G_1-G_3^2 4^{-1}G_2^{-1} \quad (16)$$

$$MSE\left(\Psi_2^*\right)_{min}=L_1-L_3^2L_2^{-1} \quad (17)$$

Biases and MSEs of Estimators $\theta_i^*,\ i=1,2$

Expressing proposed estimators $\theta_i^*,\ i=1,2$ in (6) and (7) in terms of error terms defined in subsection 3.1 to obtain (18) and (19) respectively.

$$\theta_1^*-S_y^2 = S_y^2e_{11}+a_1\left(\frac{1}{2}\left(e_0-e_6\right)+\frac{1}{4}\left(e_0^2-e_6^2\right)\right) -a_1^2\frac{1}{8}\left(e_0^2+e_6^2-2e_0e_6\right) \quad (18)$$

$$\theta_2^*-S_y^2 = S_y^2e_{11}+a_2\left(\frac{1}{2}\left(e_3-e_{12}\right)+\frac{1}{4}\left(e_{12}^2-e_3^2\right)\right) -a_1^2\frac{1}{8}\left(e_3^2+e_{12}^2-2e_3e_{12}\right) \quad (19)$$

Take expectation of both sides of (18) and (19) to obtain the biases of $\theta_i^*,\ i=1,2$ as in (20) and (21) respectively.

$$Bias\left(\theta_1^*\right)= a_1\frac{C_z^2}{2}\left(\kappa_6+\kappa_7\right)-a_1^2\frac{C_z^2}{8}\left(\kappa_6-\kappa_7\right) \quad (20)$$

$$Bias\left(\theta_2^*\right)= a_2\frac{1}{4}\left(\lambda_{004}-1\right)\left(\kappa_7+\kappa_6\right)- a_1^2\frac{1}{8}\left(\lambda_{004}-1\right)\left(\kappa_6-\kappa_7\right) \quad (21)$$

Squaring both sides of (18), (19) and take expectation of the results to obtain the MSEs of $\theta_i^*,\ i=1,2$ as in (22) and (23) respectively.

$$MSE\left(\theta_1^*\right)= \kappa_3 S_y^4\left(\lambda_{400}-1\right)+a_1^2\frac{C_z^2}{4}\left(\kappa_6-\kappa_7\right) +a_1 S_y^2\kappa_3\left(\lambda_{400}-1\right) \quad (22)$$

$$MSE\left(\theta_2^*\right)= \kappa_3 S_y^4\left(\lambda_{400}-1\right)+a_2^2\frac{\left(\lambda_{004}-1\right)\left(\kappa_6-\kappa_7\right)}{4} +a_2 S_y^2\left(\lambda_{202}-1\right)\left(\kappa_7-\kappa_6\right) \quad (23)$$

Differentiate (22) and (23) partially with respect to a_1 and a_2 respectively, equate the result to zero and solve for a_1 and a_2, the expression for the optimum values of a_1 and a_2 are obtained as in (24) and (25) respectively.

$$a_{1(opt)}=\frac{-2S_y^2\kappa_3\left(\lambda_{400}-1\right)}{C_z^2\left(\kappa_6-\kappa_7\right)} \quad (24)$$

$$a_{2(opt)}=\frac{-2S_y^2\left(\lambda_{202}-1\right)\left(\kappa_7-\kappa_6\right)}{\left(\lambda_{004}-1\right)\left(\kappa_6-\kappa_7\right)} \quad (25)$$

Substitute $a_{1(opt)}$ and $a_{2(opt)}$ in (22) and (23), respectively, the minimum MSE of $\theta_i^*,\ i=1,2$ are obtained as in (26) and (27) respectively.

$$MSE\left(\theta_1^*\right)_{\min}= S_y^4\kappa_3\left(\lambda_{400}-1\right)-\frac{S_y^4\kappa_3^2\left(\lambda_{400}-1\right)^2}{C_z^2\left(\kappa_6-\kappa_7\right)} \quad (26)$$

$$MSE\left(\theta_2^*\right)_{\min}= S_y^4\kappa_3\left(\lambda_{400}-1\right)-\frac{S_y^4\left(\lambda_{202}-1\right)^2\left(\kappa_7-\kappa_6\right)^2}{\left(\lambda_{004}-1\right)\left(\kappa_6-\kappa_7\right)} \quad (27)$$

Biases and MSEs of Estimators $\Theta_i^*,\ i=1,2,3,4$

Using the results of subsections 3.1.1 and 3.1.2, the biases of the estimators $\Theta_i^*,\ i=1,2,3,4$ are obtained as in (28), (29), (30) and (31) respectively.

$$Bias\left(\Theta_1^*\right)= \varpi_1 Bias\left(\Psi_1^*\right)+\left(1-\varpi_1\right)Bias\left(\theta_1^*\right), \quad (28)$$

$$Bias\left(\Theta_2^*\right)= \varpi_2 Bias\left(\Psi_1^*\right)+\left(1-\varpi_2\right)Bias\left(\theta_2^*\right), \quad (29)$$

$$Bias\left(\Theta_3^*\right)= \varpi_1 Bias\left(\Psi_2^*\right)+\left(1-\varpi_1\right)Bias\left(\theta_1^*\right), \quad (30)$$

$$Bias\left(\Theta_4^*\right)= \varpi_2 Bias\left(\Psi_2^*\right)+\left(1-\varpi_2\right)Bias\left(\theta_2^*\right), \quad (31)$$

And MSEs of the estimators $\Theta_i^*,\ i=1,2,3,4$ are obtained as in (32), (33), (34) and (35) respectively.

$$MSE\left(\Theta_1^*\right)= \varpi_1^2 MSE\left(\Psi_1^*\right)+\left(1-\varpi_1\right)^2 MSE\left(\theta_1^*\right), \quad (32)$$

$$MSE\left(\Theta_2^*\right)= \varpi_2^2 MSE\left(\Psi_1^*\right)+\left(1-\varpi_2\right)^2 MSE\left(\theta_2^*\right), \quad (33)$$

$$MSE\left(\Theta_3^*\right)= \varpi_1^2 MSE\left(\Psi_2^*\right)+\left(1-\varpi_1\right)^2 MSE\left(\theta_1^*\right), \quad (34)$$

$$MSE\left(\Theta_4^*\right)= \varpi_2^2 MSE\left(\Psi_2^*\right)+\left(1-\varpi_2\right)^2 MSE\left(\theta_2^*\right), \quad (35)$$

Differentiate (32) to (35) partially with respect to $\varpi_i,\ i=1,2,3,4$, equate result to zero and solve for $\varpi_i,\ i=1,2,3,4$, we obtained,

$$\varpi_{1(opt)}=\frac{MSE\left(\theta_1^*\right)_{\min}}{MSE\left(\Psi_1^*\right)_{min}+MSE\left(\theta_1^*\right)_{\min}}, \quad (36)$$

$$\varpi_{2(opt)}=\frac{MSE\left(\theta_2^*\right)_{\min}}{MSE\left(\Psi_1^*\right)_{min}+MSE\left(\theta_2^*\right)_{\min}}, \quad (37)$$

$$\varpi_{3(opt)}=\frac{MSE\left(\theta_1^*\right)_{\min}}{MSE\left(\Psi_2^*\right)_{min}+MSE\left(\theta_1^*\right)_{\min}}, \quad (38)$$

$$\varpi_{4(opt)}=\frac{MSE\left(\theta_2^*\right)_{\min}}{MSE\left(\Psi_2^*\right)_{min}+MSE\left(\theta_2^*\right)_{\min}}, \quad (39)$$

Substitute (36) in (32), (37) in (33), (38) in (34), (39) in (35), the minimum MSEs of $\Theta_i^*,\ i=1,2,3,4$ denoted by $MSE\left(\Theta_i^*\right)_{\min}$, $i=1,2,3,4$ is obtained as in (40), (41), (42), and (43) respectively.

$$MSE\left(\Theta_1^*\right)_{\min}=\frac{MSE\left(\Psi_1^*\right)_{min}MSE\left(\theta_1^*\right)_{\min}}{MSE\left(\Psi_1^*\right)_{min}+MSE\left(\theta_1^*\right)_{\min}} \quad (40)$$

$$MSE\left(\Theta_2^*\right)_{\min}=\frac{MSE\left(\Psi_1^*\right)_{min}MSE\left(\theta_2^*\right)_{\min}}{MSE\left(\Psi_1^*\right)_{min}+MSE\left(\theta_2^*\right)_{\min}} \quad (41)$$

$$MSE\left(\Theta_3^*\right)_{\min}=\frac{MSE\left(\Psi_2^*\right)_{min}MSE\left(\theta_1^*\right)_{\min}}{MSE\left(\Psi_2^*\right)_{min}+MSE\left(\theta_1^*\right)_{\min}} \quad (42)$$

$$MSE\left(\Theta_4^*\right)_{\min}=\frac{MSE\left(\Psi_2^*\right)_{min}MSE\left(\theta_2^*\right)_{\min}}{MSE\left(\Psi_2^*\right)_{min}+MSE\left(\theta_2^*\right)_{\min}} \quad (43)$$

Effect of Measurement Error on the Efficiency of $\Theta_i^*,\ i=1,2,3,4$

Let the true values of X, Y and Z respectively be x_i, y_i and z_i while the observed values be x_i^*, y_i^* and z_i^*. The measurement errors on X, Y and Z are defined as $u_i=x_i-x_i^*$, $v_i=y_i-y_i^*$ and $w_i=z_i-z_i^*$ respectively such that $u_i\in U\sim N\left(0,S_u^2\right), v_i\in V\sim N\left(0,S_v^2\right), w_i\in W\sim N\left(0,S_w^2\right)$ and pairs of X, Y, Z, U, V and W are uncorrelated. Here we considered two cases.

Case I: Auxiliary variable Z assumed to be free measurement errors, that is, $w_i=0$. Then, the joint moment about that mean is given as

$$\tau_{pqr}=\frac{1}{N-1}\sum_{i=1}^{N}v_i^p u_i^q\left(z_i-\bar{Z}\right)^s \quad (44)$$

Take into consideration the effects of the measurement errors U and V, the expression for the minimum MSE of the proposed estimators $\Theta_i^*,\ i=1,2,3,4$ are obtained as in (45), (46), (47) and (48).

$$MSE\left(\Theta_1^*\right)_{\min.m(I)}=\frac{MSE\left(\Psi_1^*\right)_{min.m(I)}MSE\left(\theta_1^*\right)_{\min.m(I)}}{MSE\left(\Psi_1^*\right)_{min.m(I)}+MSE\left(\theta_1^*\right)_{\min.(I)}} \quad (45)$$

$$MSE\left(\Theta_2^*\right)_{\min.m(I)}=\frac{MSE\left(\Psi_1^*\right)_{min.m(I)}MSE\left(\theta_2^*\right)_{\min.m(I)}}{MSE\left(\Psi_1^*\right)_{min.m(I)}+MSE\left(\theta_2^*\right)_{\min.(I)}} \quad (46)$$

$$MSE\left(\Theta_3^*\right)_{\min.m(I)}=\frac{MSE\left(\Psi_2^*\right)_{min.m(I)}MSE\left(\theta_1^*\right)_{\min.m(I)}}{MSE\left(\Psi_2^*\right)_{min.m(I)}+MSE\left(\theta_1^*\right)_{\min.(I)}} \quad (47)$$

$$MSE\left(\Theta_1^*\right)_{\min.m(I)}=\frac{MSE\left(\Psi_2^*\right)_{min.m(I)}MSE\left(\theta_2^*\right)_{\min.m(I)}}{MSE\left(\Psi_2^*\right)_{min.m(I)}+MSE\left(\theta_2^*\right)_{\min.(I)}} \quad (48)$$

where $MSE\left(\Psi_1^*\right)_{min.m(I)}=G_{1.m(I)}-\frac{G_{3.m(I)}^2}{4G_{2.m(I)}}$,

$MSE\left(\Psi_2^*\right)_{min.m(I)}=L_{1.m(I)}-\frac{L_{3.m(I)}^2}{L_{2,m(I)}}$,

$$G_{1m(I)}=\left(S_y^4+S_v^4\right)\kappa_2\left(\lambda_{400}-1\right)+\frac{1}{4}\left(\lambda_{004}-1\right)\left(\kappa_5-\kappa_7\right)+\left(S_y^2+S_v^2\right)\left(\lambda_{202}-1\right)\left(\kappa_7-\kappa_5\right)$$

$$G_{2m(I)}=\frac{1}{4}\left(\left(\lambda_{004}-1\right)\left(\kappa_5-\kappa_7\right)+C_z^2\left(\kappa_5-\kappa_7\right)\right)+\left(S_x^2+S_u^2\right)\left(\kappa_2-\kappa_1\right)+\frac{1}{2}C_z\lambda_{003}\left(\kappa_7-\kappa_5\right)+\bar{X}\sqrt{C_x^2+\frac{S_u^2}{\bar{X}^2}}\lambda_{012}\left(\kappa_5-\kappa_1\right)+\bar{X}C_{xz}\left(\kappa_1-\kappa_5\right),$$

$$G_{3m(I)}=\left(S_y^2+S_v^2\right)\begin{pmatrix}\left(\lambda_{202}-1\right)\left(\kappa_7-\kappa_5\right)+C_z\lambda_{201}\left(\kappa_5-\kappa_7\right)\\+2\bar{X}\sqrt{C_x^2+\frac{S_u^2}{\bar{X}^2}}\lambda_{201}\left(\kappa_1-\kappa_2\right)\end{pmatrix}+\frac{1}{2}\left(\lambda_{004}-1\right)\left(\kappa_5-\kappa_7\right)+\frac{1}{2}C_z\lambda_{003}\left(\kappa_7-\kappa_5\right)+\bar{X}\sqrt{C_x^2+\frac{S_u^2}{\bar{X}^2}}\lambda_{012}\left(\kappa_5-\kappa_1\right),$$

.

$$L_{1m(I)} = \left(S_y^4 + S_v^4\right)\kappa_2\left(\lambda_{400} - 1\right) + \frac{1}{4}\left(\lambda_{004} - 1\right)\left(\kappa_7 + \kappa_5\right)$$

$$L_{2m(I)} = \frac{1}{4}\left(\lambda_{004} - 1\right)\left(\kappa_5 + \kappa_4\right) - \left(S_x^4 + S_u^4\right)\left(\lambda_{040} - 1\right)\left(\kappa_2 - 2\kappa_1\right) + \left(S_x^2 + S_u^2\right)\kappa_1\left(\lambda_{040} - 1\right) - \frac{1}{2}\kappa_5\sqrt{C_x^2 + \frac{S_u^2}{\bar{X}^2}}\lambda_{012} + \left(S_x^2 + S_u^2\right)\left(\lambda_{022} - 1\right)\left(\kappa_2 - \kappa_5\right)$$

$$L_{3m(I)} = \left(S_y^2 + S_v^2\right)\left(\lambda_{202} - 1\right)\left(\kappa_5 - \kappa_4\right) + 2\left(S_x^2 + S_u^2\right)\left(S_y^2 + S_v^2\right)\left(\lambda_{220} - 1\right)\left(\kappa_1 - \kappa_2\right) + \frac{1}{2}\left(\lambda_{004} - 1\right)\left(\kappa_4 - \kappa_5\right) + S_x^2\left(\lambda_{022} - 1\right)\left(\kappa_5 - \kappa_2\right),$$

$$MSE\left(\theta_1^*\right)_{\min(I)} = \kappa_3\left(S_y^4 + S_v^4\right)\left(\lambda_{400} - 1\right) + a_1^2\frac{C_z^2}{4}\left(\kappa_6 - \kappa_7\right) + a_1\left(S_y^2 + S_v^2\right)\kappa_3\left(\lambda_{400} - 1\right)$$

$$MSE\left(\theta_2^*\right)_{\min(I)} = \kappa_3\left(S_y^4 + S_v^4\right)\left(\lambda_{400} - 1\right) + a_2^2\frac{\left(\lambda_{004} - 1\right)\left(\kappa_6 - \kappa_7\right)}{4} + a_2\left(S_y^2 + S_v^2\right)\left(\lambda_{202} - 1\right)\left(\kappa_7 - \kappa_6\right)$$

Case II: Auxiliary variable Z assumed to be characterized by measurement errors. Therefore, the joint moment about that mean is given as

$$\varpi_{pqs} = \frac{1}{N-1}\sum_{i=1}^{N} v_i^p u_i^q w_i^s \tag{49}$$

Take into consideration the effects of the measurement errors U, V and W, the expression for the minimum MSE of the proposed estimators Φ_j, $j = 1, 2, 3, 4$ are obtained as in (50).

$$MSE\left(\Theta_i^*\right)_{\min.m(II)} = \frac{MSE\left(\Psi_j^*\right)_{min.m(II)} MSE\left(\theta_j^*\right)_{\min.m(II)}}{MSE\left(\Psi_j^*\right)_{min.m(II)} + MSE\left(\theta_j^*\right)_{\min.(II)}} \tag{50}$$

$$MSE\left(\Theta_1^*\right)_{\min.m(II)} = \frac{MSE\left(\Psi_1^*\right)_{min.m(II)} MSE\left(\theta_1^*\right)_{\min.m(II)}}{MSE\left(\Psi_1^*\right)_{min.m(II)} + MSE\left(\theta_1^*\right)_{\min.(II)}} \tag{51}$$

$$MSE\left(\Theta_2^*\right)_{\min.m(II)} = \frac{MSE\left(\Psi_1^*\right)_{min.m(II)} MSE\left(\theta_2^*\right)_{\min.m(II)}}{MSE\left(\Psi_1^*\right)_{min.m(II)} + MSE\left(\theta_2^*\right)_{\min.(II)}} \tag{52}$$

$$MSE\left(\Theta_3^*\right)_{\min.m(II)} = \frac{MSE\left(\Psi_2^*\right)_{min.m(II)} MSE\left(\theta_1^*\right)_{\min.m(II)}}{MSE\left(\Psi_2^*\right)_{min.m(II)} + MSE\left(\theta_1^*\right)_{\min.(II)}} \tag{53}$$

$$MSE\left(\Theta_4^*\right)_{\min.m(II)} = \frac{MSE\left(\Psi_2^*\right)_{min.m(II)} MSE\left(\theta_2^*\right)_{\min.m(II)}}{MSE\left(\Psi_2^*\right)_{min.m(II)} + MSE\left(\theta_2^*\right)_{\min.(II)}} \tag{54}$$

where $MSE\left(\Psi_1^*\right)_{min.m(II)} = G_{1.m(II)} - \frac{G_{3.m(II)}^2}{4G_{2.m(II)}}$,

$$MSE\left(\Psi_2^*\right)_{min.m(II)} = L_{1.m(II)} - \frac{L_{3.m(II)}^2}{L_{2.m(II)}},$$

$$G_{1m(II)} = G_{1m(I)}$$

$$G_{2m(II)} = \frac{1}{4}\left(\left(\lambda_{004} - 1\right)\left(\kappa_5 - \kappa_7\right) + C_z^2 + \frac{S_w^2}{\bar{Z}^2}\left(\kappa_5 - \kappa_7\right)\right) + \left(S_x^2 + S_u^2\right)\left(\kappa_2 - \kappa_1\right) + \frac{1}{2}\sqrt{C_z^2 + \frac{S_w^2}{\bar{Z}^2}}\lambda_{003}\left(\kappa_7 - \kappa_5\right) + \bar{X}\sqrt{C_x^2 + \frac{S_u^2}{\bar{X}^2}}\lambda_{012}\left(\kappa_5 - \kappa_1\right) + \bar{X}C_{xz}\left(\kappa_1 - \kappa_5\right),$$

,

$$G_{3m(II)} = \left(S_y^2+S_v^2\right)\begin{pmatrix}\left(\lambda_{202}-1\right)\left(\kappa_7-\kappa_5\right)+\sqrt{C_z^2+\frac{S_w^2}{\bar{Z}^2}}\lambda_{201} \\ \left(\kappa_5-\kappa_7\right)+2\bar{X}\sqrt{C_x^2+\frac{S_u^2}{\bar{X}^2}}\lambda_{201}\left(\kappa_1-\kappa_2\right)\end{pmatrix},$$

$$\frac{1}{2}\left(\lambda_{004}-1\right)\left(\kappa_5-\kappa_7\right)+\frac{1}{2}\sqrt{C_z^2+\frac{S_w^2}{\bar{Z}^2}}\lambda_{003}\left(\kappa_7-\kappa_5\right)$$

$$+\bar{X}\sqrt{C_x^2+\frac{S_u^2}{\bar{X}^2}}\lambda_{012}\left(\kappa_5-\kappa_1\right),$$

.

$$L_{1m(II)}=L_{1m(I)}\ L_{2m(II)}=L_{2m(I)},\ L_{3m(II)}=L_{3m(I)}.$$

$$MSE\left(\theta_1^*\right)_{\min(II)}=MSE\left(\theta_1^*\right)_{\min(I)},$$

$$MSE\left(\theta_2^*\right)_{\min(II)}=MSE\left(\theta_2^*\right)_{\min(I)}$$

Data and procedure for Empirical studies on the Efficiency Comparisons of $\Theta_i,\ i=1,2,3,4$

To evaluate the efficiency of the proposed estimators $\Theta_i^*,\ i=1,2,3,4$, the ARBs, MSEs and PREs of $\Theta_i^*,\ i=1,2,3,4$ were compare to that of estimator Θ_0 which is defined as follows:

$$\Theta_0=\left(1-\varpi_0\right)s_{y_{u-\eta_3}}^2+\varpi_0 s_{y_{m-\eta_2}}^2\frac{s_{x_{n-\eta_1}}^2}{s_{x_{m-\eta_2}}^2} \qquad (55)$$

where $\varpi_0\left(0\le\varpi_0\le 1\right)$ is an unknown quantity to be obtained by minimizing the MSE of Θ_0 .The optimum expression for ϖ_0 and minimum MSE of Θ_0 up to second degree approximation in the absence of measurement errors are obtained as in (56) and (57) respectively.

$$\varpi_{0(opt)}=\frac{\operatorname{var}\left(\theta_0\right)}{MSE\left(\Psi_0\right)+\operatorname{var}\left(\theta_0\right)} \qquad (56)$$

$$MSE\left(\Phi_0\right)_{\min}=\frac{MSE\left(T_0\right)\operatorname{var}\left(t_0\right)}{MSE\left(T_0\right)+\operatorname{var}\left(t_0\right)} \qquad (57)$$

where

$$MSE\left(\Psi_0\right)= S_y^4\left(\kappa_2\left(\lambda_{400}-1\right)+\kappa_9\begin{pmatrix}\left(\lambda_{400}-1\right) \\ +\left(\lambda_{040}-1\right)-2\left(\lambda_{220}-1\right)\end{pmatrix}\right)$$

$$\operatorname{var}\left(\theta_0\right)=S_y^4\kappa_3\left(\lambda_{400}-1\right), \qquad \theta_0=s_{y_{u-\eta_3}}^2,$$

$$\Psi_0=s_{y_{m-\eta_2}}^2 s_{x_{n-\eta_1}}^2 s_{x_{m-\eta_2}}^{-2}$$

The optimum expression for θ_0 and minimum MSE of Φ_0 up to second degree approximation in the presence of measurement errors are obtained as in (58) and (9) respectively

$$\varpi_{0(opt)m}=\frac{\operatorname{var}\left(\theta_0\right)_m}{MSE\left(\Psi_0\right)_m+\operatorname{var}\left(\theta_0\right)_m} \qquad (58)$$

$$MSE\left(\Theta_0\right)_{\min(m)}=\frac{MSE\left(\Psi_0\right)_m\operatorname{var}\left(\theta_0\right)_m}{MSE\left(\Psi_0\right)_m+\operatorname{var}\left(\theta_0\right)_m} \qquad (59)$$

where

$$MSE\left(T_0\right)_m= \left(S_y^4+S_v^4\right)\left(\kappa_2\left(\lambda_{400}-1\right)+\kappa_9\begin{pmatrix}\left(\lambda_{400}-1\right) \\ +\left(\lambda_{040}-1\right)-2\left(\lambda_{220}-1\right)\end{pmatrix}\right),$$

$$\operatorname{var}\left(\theta_0\right)_m=\kappa_3\left(S_y^4+S_v^4\right)\left(\lambda_{400}-1\right).$$

To assess the performance of the proposed estimators $\Theta_i,\ i=1,2,3,4$ with respect to Θ_0, absolute relative bias (ARB), MSE and percentage relative efficiency (PRE) of the estimators were computed using (60), (61) and (62) respectively.

$$ARB\left(\Theta_i\right)=\frac{\left|E\left(\Theta_i-S_y^2\right)\right|}{S_y^2},\quad i=0,1,2,3,4 \qquad (60)$$

$$MSE\left(\Theta_i\right)=E\left(\Theta_i-S_y^2\right)^2,\quad i=0,1,2,3,4 \qquad (61)$$

$$PRE\left(\Theta_i\right)=\frac{E\left(\Theta_0-S_y^2\right)^2}{E\left(\Theta_i-S_y^2\right)^2}\times 100,\ i=0,1,2,3,4 \qquad (62)$$

where
$E\left(\Theta_i - S_y^2\right) = \frac{1}{M}\sum_{k=1}^{M}\left(\Theta_{ik} - S_y^2\right),\ E\left(\Theta_i - S_y^2\right)^2 = \frac{1}{M}\sum_{k=1}^{M}\left(\Theta_{ik} - S_y^2\right)^2$

Artificial populations of size $N = 10,000$ from which SRSWOR of size $n = m + u = 1000$ were drawn
$(m = 900, u = 100)$,
$(m = 800, u = 200)$,
$(m = 700, u = 300)$,
$(m = 600, u = 400)$,
$(m = 500, u = 500)$
times are simulated from multivariate normal distribution with parameters
$\mu_X = 20, \mu_Y = 30, \mu_Z = 50, \sigma_X^2 = 10, \sigma_Y^2 = 9, \sigma_Z^2 = 10, \mu_U = \mu_V = \mu_W = 0, \sigma_U^2 = \sigma_V^2 = \sigma_W^2 = 1$
for different probabilities p_1, p_2, p_3 of non-response in the presence of measurement errors and ABR, MSE and PRE of the estimators $\Theta_1^*,\ i = 1, 2, 3, 4$ were computed for each estimators using (60), (61) and (62) respectively.

RESULTS AND DISCUSSION

The result of ARB, MSE, and PRE of $\Theta_0, \Theta_i and \Theta_i^*, i = 0, 1, 2, 3, 4$ in different match (m) and un-match (u) value at different probabilities value
$(P_1 = 0.1, P_2 = 0.1, P_3 = 0.1)$,
$(P_1 = 0.15, P_2 = 0.15, P_3 = 0.15)$,
$(P_1 = 0.1, P_2 = 0.15, P_3 = 0.2)$,
$(P_1 = 0.15, P_2 = 0.2, P_3 = 0.25)$, ,
$(P_1 = 0.25, P_2 = 0.25, P_3 = 0.1)$
$(P_1 = 0.2, P_2 = 0.2, P_3 = 0.2)$,
$(P_1 = 0.25, P_2 = 0.25, P_3 = 0.25)$'

Table 1: ARB, MSE, and PRE of $\Theta_0, \Theta_i and \Theta_i^*, i = 0, 1, 2, 3, 4$ when n= 1000, m=900 and u=100

Estimators	ARB	MSEs	PREs	Estimators	ARB	MSEs	PREs
$P_1 = 0.1, P_2 = 0.1, P_3 = 0.1$				$P_1 = 0.2, P_2 = 0.2, P_3 = 0.2$			
Θ_0	0.0409585	1.71409	100	Θ_0	0.0118691	1.75085	100
Audu *et al.*(2025)				Audu *et al.*(2025)			
Θ_1	-0.1002	0.61739	276.9868	Θ_1	-0.10899	1.70029	100.58
Θ_2	-0.12042	0.56406	303.1748	Θ_2	-0.22001	1.56925	108.97
Θ_3	-0.00311	0.18613	918.7808	Θ_3	-0.00399	0.21031	813.11
Θ_4	-0.00403	0.18776	910.7824	Θ_4	-0.0048	0.21207	806.4
Proposed Estimators when $n \subset n_1$				Proposed Estimators when $n \subset n_1$			
Θ_1^*	0.1672799	0.58632	292.35	Θ_1^*	0.1788964	1.58294	110.61
Θ_2^*	0.16445	0.53698	319.21	Θ_2^*	0.1781459	1.46197	119.76
Θ_3^*	0.0109015	0.18773	913.06	Θ_3^*	0.0145059	0.21208	825.56
Θ_4^*	0.0126006	0.18921	905.92	Θ_4^*	0.0157653	0.2137	819.3
Proposed Estimators when $n \subset u$				Proposed Estimators when $n \subset u$			
Θ_1^*	0.1629043	0.61739	276.99	Θ_1^*	0.17634	1.70029	100.58
Θ_2^*	0.1603442	0.56406	303.18	Θ_2^*	0.17563	1.56925	108.97

Θ_3^*	0.007207	0.18613	918.76	Θ_3^*	0.0087	0.21031	813.13
Θ_4^*	0.0093088	0.18776	910.79	Θ_4^*	0.01042	0.21207	806.38
$P_1 = 0.15, P_2 = 0.15, P_3 = 0.15$				$P_1 = 0.25, P_2 = 0.25, P_3 = 0.25$			
Θ_0	0.0129778	1.88418	100	Θ_0	0.0188192	1.92101	100
Audu *et al.*(2025)				Audu *et al.*(2025)			
Θ_1	-0.10491	0.99374	172.09	Θ_1	-0.1172	1.54484	110.7
Θ_2	-0.21189	0.90511	188.94	Θ_2	0.2293	1.53763	111.22
Θ_3	-0.0195	0.19961	856.73	Θ_3	-0.00499	0.22436	762.19
Θ_4	-0.02047	0.20139	849.14	Θ_4	-0.00575	0.22628	755.74
Proposed Estimators when $n \subset n_1$				Proposed Estimators when $n \subset n_1$			
Θ_1^*	0.2256037	1.05119	179.24	Θ_1^*	0.1637034	1.68207	114.21
Θ_2^*	0.2243666	0.95612	197.07	Θ_2^*	0.1635982	1.67408	114.75
Θ_3^*	0.0489204	0.20109	936.98	Θ_3^*	0.0158568	0.2251	853.4
Θ_4^*	0.0504213	0.2027	929.54	Θ_4^*	0.016958	0.22687	846.74
Proposed Estimators when $n \subset u$				Proposed Estimators when $n \subset u$			
Θ_1^*	0.22407	0.99374	172.09	Θ_1^*	0.18504	1.54484	110.7
Θ_2^*	0.22272	0.90511	188.94	Θ_2^*	0.18491	1.53763	111.22
Θ_3^*	0.04366	0.19961	856.72	Θ_3^*	0.01053	0.22436	762.21
Θ_4^*	0.04561	0.20139	849.14	Θ_4^*	0.01209	0.22628	755.74

Table 2: ARB, MSE, and PRE of $\Theta_0, \Theta_i and \Theta_i^*, i = 0,1,2,3,4$ when n= 1000, m=800 and u=200

Estimators	ARB	MSEs	PREs	Estimators	ARB	MSEs	PREs
$P_1 = 0.1, P_2 = 0.1, P_3 = 0.1$				$P_1 = 0.2, P_2 = 0.2, P_3 = 0.2$			
Θ_0	0.0119399	0.83	100	Θ_0	0.0159472	0.89318	100
Audu *et al.*(2025)				Audu *et al.*(2025)			
Θ_1	-0.05641	0.44689	382.66	Θ_1	-0.0595	0.97758	174.93
Θ_2	-0.07027	0.3403	502.53	Θ_2	-0.12301	0.68725	248.83
Θ_3	-0.01058	0.1916	892.52	Θ_3	-0.02936	0.21489	795.8
Θ_4	-0.00718	0.19595	872.74	Θ_4	-0.02338	0.21594	791.94
Proposed Estimators when $n \subset n_1$				Proposed Estimators when $n \subset n_1$			

Θ_1^*	0.1443226	0.45187	183.68	Θ_1^*	0.1610088	0.97227	91.87
Θ_2^*	0.1218276	0.3433	241.77	Θ_2^*	0.151003	0.68368	130.64
Θ_3^*	0.0337938	0.1969	421.53	Θ_3^*	0.0749597	0.22009	405.82
Θ_4^*	0.0217908	0.19984	415.33	Θ_4^*	0.0582802	0.21959	406.75
Proposed Estimators when $n \subset u$				Proposed Estimators when $n \subset u$			
Θ_1^*	0.1421	0.44689	382.66	Θ_1^*	0.1579372	0.97758	174.93
Θ_2^*	0.12045	0.3403	502.52	Θ_2^*	0.1483769	0.68725	248.83
Θ_3^*	0.02417	0.1916	892.53	Θ_3^*	0.063326	0.21489	795.8
Θ_4^*	0.01622	0.19595	872.72	Θ_4^*	0.0503079	0.21594	791.93
$P_1 = 0.15, P_2 = 0.15, P_3 = 0.15$				$P_1 = 0.25, P_2 = 0.25, P_3 = 0.25$			
Θ_0	0.0145506	0.95725	100	Θ_0	0.0041633	0.82614	100
Audu *et al.*(2025)				Audu *et al.*(2025)			
Θ_1	-0.05511	0.83414	205.01	Θ_1	-0.07085	1.75991	97.17
Θ_2	-0.10329	0.57029	299.86	Θ_2	-0.23625	1.22768	139.3
Θ_3	-0.00469	0.20142	849.02	Θ_3	-0.02496	0.24917	686.31
Θ_4	-0.00035	0.20499	834.23	Θ_4	-0.02141	0.24591	695.41
Proposed Estimators when $n \subset n_1$				Proposed Estimators when $n \subset n_1$			
Θ_1^*	0.1566062	0.79513	120.39	Θ_1^*	0.2225808	1.66039	49.76
Θ_2^*	0.1396259	0.54795	174.7	Θ_2^*	0.2153818	1.16419	70.96
Θ_3^*	0.0239331	0.20559	465.61	Θ_3^*	0.0628289	0.25377	325.55
Θ_4^*	0.0104666	0.20794	460.35	Θ_4^*	0.0530078	0.24915	331.58
Proposed Estimators when $n \subset u$				Proposed Estimators when $n \subset u$			
Θ_1^*	0.15294	0.83414	205.01	Θ_1^*	0.21995	1.75991	97.17
Θ_2^*	0.13678	0.57029	299.86	Θ_2^*	0.21322	1.22768	139.29
Θ_3^*	0.01044	0.20142	849.02	Θ_3^*	0.05	0.24917	686.31
Θ_4^*	0.00077	0.20499	834.23	Θ_4^*	0.04318	0.24591	695.41

Table 3: ARB, MSE, and PRE of $\Theta_0, \Theta_i and \Theta_i^*, i = 0,1,2,3,4$ when n= 1000, m=700 and u=300

Estimators	ARB	MSEs	PREs	Estimators	ARB	MSEs	PREs
$P_1 = 0.1, P_2 = 0.1, P_3 = 0.1$				$P_1 = 0.2, P_2 = 0.2, P_3 = 0.2$			

Θ_0	0.0038837	0.57666	100	Θ_0	0.0093488	0.58159	100
Audu *et al.*(2025)				Audu *et al.*(2025)			
Θ_1	-0.04125	0.27677	617.88	Θ_1	-0.04763	0.31112	549.66
Θ_2	-0.03981	0.35628	479.99	Θ_2	-0.12563	0.61108	279.85
Θ_3	-0.01565	0.24454	699.3	Θ_3	0.02072	0.35935	475.89
Θ_4	-0.04821	0.25013	683.69	Θ_4	-0.08733	0.35841	477.13
Proposed Estimators when $n \subset n_1$				Proposed Estimators when $n \subset n_1$			
Θ_1^*	0.0209405	0.27732	207.94	Θ_1^*	0.0407402	0.31068	187.2
Θ_2^*	0.0672876	0.35692	161.57	Θ_2^*	0.1626718	0.58672	99.13
Θ_3^*	0.0353175	0.2471	233.37	Θ_3^*	0.0414363	0.35802	162.45
Θ_4^*	0.1107878	0.25899	222.66	Θ_4^*	0.1637314	0.36419	159.69
Proposed Estimators when $n \subset u$				Proposed Estimators when $n \subset u$			
Θ_1^*	0.02176	0.27677	617.87	Θ_1^*	0.04079	0.31112	549.66
Θ_2^*	0.06669	0.35628	479.98	Θ_2^*	0.16071	0.61108	279.85
Θ_3^*	0.03165	0.24454	699.31	Θ_3^*	0.03456	0.35935	475.88
Θ_4^*	0.09639	0.25013	683.68	Θ_4^*	0.14586	0.35841	477.13
$P_1 = 0.15, P_2 = 0.15, P_3 = 0.15$				$P_1 = 0.25, P_2 = 0.25, P_3 = 0.25$			
Θ_0	0.0311265	0.60054	100	Θ_0	0.0671268	0.54582	100
Audu *et al.*(2025)				Audu *et al.*(2025)			
Θ_1	-0.0488	0.29019	589.3	Θ_1	-0.05346	0.63885	267.68
Θ_2	-0.09791	0.45357	377.03	Θ_2	-0.14426	0.81406	210.07
Θ_3	-0.02094	0.27958	611.65	Θ_3	0.13581	1.06979	159.85
Θ_4	-0.07221	0.30404	562.46	Θ_4	-0.10045	0.42849	399.09
Proposed Estimators when $n \subset n_1$				Proposed Estimators when $n \subset n_1$			
Θ_1^*	0.0394892	0.28987	207.18	Θ_1^*	0.1477776	0.64012	85.27
Θ_2^*	0.1468308	0.45746	131.28	Θ_2^*	0.1602696	0.81658	66.84
Θ_3^*	0.0401268	0.27997	214.5	Θ_3^*	0.1445168	1.04613	52.18
Θ_4^*	0.1451775	0.31984	187.76	Θ_4^*	0.1670223	0.42922	127.17
Proposed Estimators when $n \subset u$				Proposed Estimators when $n \subset u$			
Θ_1^*	0.04018	0.29019	589.3	Θ_1^*	0.14816	0.63885	267.68

Θ_2^*	0.14539	0.45357	377.03	Θ_2^*	0.15989	0.81406	210.07
Θ_3^*	0.03959	0.27958	611.66	Θ_3^*	0.1313	1.06979	159.85
Θ_4^*	0.13095	0.30404	562.46	Θ_4^*	0.15345	0.42849	399.1

Table 4: ARB, MSE, and PRE of $\Theta_0, \Theta_i and \Theta_i^*, i = 0,1,2,3,4$ when n= 1000, m=600 and u=400

Estimators	ARB	MSEs	PREs	Estimators	ARB	MSEs	PREs
$P_1 = 0.1, P_2 = 0.1, P_3 = 0.1$				$P_1 = 0.2, P_2 = 0.2, P_3 = 0.2$			
Θ_0	0.015454	0.42739	100	Θ_0	0.0762824	0.43881	100
Audu *et al.*(2025)				Audu *et al.*(2025)			
Θ_1	0.00026	0.33125	516.25	Θ_1	-0.03985	0.42298	404.3
Θ_2	-0.03488	0.35921	476.07	Θ_2	-0.09436	0.49994	342.06
Θ_3	-0.62529	10.68724	16	Θ_3	-1.03588	21.50482	7.95
Θ_4	-0.27841	2.21421	77.23	Θ_4	-0.38351	2.78369	61.43
Proposed Estimators when $n \subset n_1$				Proposed Estimators when $n \subset n_1$			
Θ_1^*	0.040631	0.33145	128.95	Θ_1^*	0.10887	0.42178	104.04
Θ_2^*	0.0610847	0.35954	118.87	Θ_2^*	0.1342181	0.49732	88.23
Θ_3^*	0.2057985	11.78588	3.63	Θ_3^*	0.2381277	19.43337	2.26
Θ_4^*	0.2013497	2.46263	17.36	Θ_4^*	0.2441407	2.52933	17.35
Proposed Estimators when $n \subset u$				Proposed Estimators when $n \subset u$			
Θ_1^*	0.03817	0.33125	516.25	Θ_1^*	0.10856	0.42298	404.3
Θ_2^*	0.05821	0.35921	476.07	Θ_2^*	0.13345	0.49994	342.06
Θ_3^*	0.19127	10.68724	16	Θ_3^*	0.22338	21.50482	7.95
Θ_4^*	0.1871	2.21421	77.23	Θ_4^*	0.22986	2.78369	61.43
$P_1 = 0.15, P_2 = 0.15, P_3 = 0.15$				$P_1 = 0.25, P_2 = 0.25, P_3 = 0.25$			
Θ_0	0.0033709	0.40358	100	Θ_0	0.0267268	0.40384	100
Audu *et al.*(2025)				Audu *et al.*(2025)			
Θ_1	-0.00749	0.33896	504.51	Θ_1	-0.04317	0.48681	351.29
Θ_2	-0.03133	0.37849	451.81	Θ_2	-0.12935	0.65715	260.23
Θ_3	-0.84272	14.6932	11.64	Θ_3	-0.92115	31.99117	5.35
Θ_4	-0.34114	2.47209	69.18	Θ_4	-0.25945	2.4436	69.98

Proposed Estimators when $n \subset n_1$				Proposed Estimators when $n \subset n_1$			
Θ_1^*	0.0298492	0.33811	119.36	Θ_1^*	0.1275162	0.48973	82.46
Θ_2^*	0.0506687	0.37684	107.1	Θ_2^*	0.1630198	0.6648	60.75
Θ_3^*	0.2297512	14.25739	2.83	Θ_3^*	0.1777069	33.49502	1.21
Θ_4^*	0.2262817	2.41251	16.73	Θ_4^*	0.1806092	2.58007	15.65
Proposed Estimators when $n \subset u$				Proposed Estimators when $n \subset u$			
Θ_1^*	0.03012	0.33896	504.51	Θ_1^*	0.12401	0.48681	351.28
Θ_2^*	0.05093	0.37849	451.82	Θ_2^*	0.15957	0.65715	260.23
Θ_3^*	0.21985	14.6932	11.64	Θ_3^*	0.16286	31.99117	5.35
Θ_4^*	0.21697	2.47209	69.18	Θ_4^*	0.16597	2.4436	69.98

Table 5: ARB, MSE, and PRE of $\Theta_0, \Theta_i and \Theta_i^*, i = 0,1,2,3,4$ when n= 1000, m=500 and u=500

Estimators	ARB	MSEs	PREs	Estimators	ARB	MSEs	PREs
$P_1 = 0.1, P_2 = 0.1, P_3 = 0.1$				$P_1 = 0.2, P_2 = 0.2, P_3 = 0.2$			
Θ_0	0.0107698	0.34322	100	Θ_0	0.0234622	0.31636	100
Audu *et al.*(2025)				Audu *et al.*(2025)			
Θ_i	-0.04024	0.38602	443.01	Θ_1	-0.00585	0.4772	358.36
Θ_2	-0.05751	0.39533	432.57	Θ_2	-0.04049	0.51028	335.13
Θ_3	-0.07649	0.37996	450.07	Θ_3	-0.06269	0.42164	405.58
Θ_4	-0.10235	0.38532	443.81	Θ_4	-0.10862	0.43342	394.56
Proposed Estimators when $n \subset n_1$				Proposed Estimators when $n \subset n_1$			
Θ_1^*	0.0810499	0.38713	88.66	Θ_1^*	0.0447878	0.47901	66.04
Θ_2^*	0.0897747	0.39655	86.55	Θ_2^*	0.0590934	0.51284	61.69
Θ_3^*	0.1221631	0.37999	90.32	Θ_3^*	0.0983212	0.42138	75.08
Θ_4^*	0.1632607	0.3851	89.12	Θ_4^*	0.1675053	0.43282	73.09
Proposed Estimators when $n \subset u$				Proposed Estimators when $n \subset u$			
Θ_1^*	0.08289	0.38602	443.01	Θ_1^*	0.04264	0.4772	358.36
Θ_2^*	0.09147	0.39533	432.57	Θ_2^*	0.05669	0.51028	335.13
Θ_3^*	0.12409	0.37996	450.07	Θ_3^*	0.09655	0.42164	405.58
Θ_4^*	0.16488	0.38532	443.81	Θ_4^*	0.16499	0.43342	394.56

$P_1 = 0.15, P_2 = 0.15, P_3 = 0.15$				$P_1 = 0.25, P_2 = 0.25, P_3 = 0.25$			
Θ_0	0.0112347	0.31015	100	Θ_0	0.0443954	0.32795	100
Audu *et al.*(2025)				Audu *et al.*(2025)			
Θ_1	-0.0623	0.40623	420.97	Θ_1	-0.01302	0.52903	323.25
Θ_2	-0.08368	0.42662	400.85	Θ_2	-0.04661	0.57184	299.05
Θ_3	-0.11036	0.38454	444.71	Θ_3	-0.07205	0.47923	356.84
Θ_4	-0.14564	0.3961	431.74	Θ_4	-0.12918	0.49861	342.97
Proposed Estimators when $n \subset n_1$				Proposed Estimators when $n \subset n_1$			
Θ_1^*	0.1202898	0.40622	76.35	Θ_1^*	0.0452149	0.52414	62.57
Θ_2^*	0.128812	0.42661	72.7	Θ_2^*	0.0605778	0.5639	58.16
Θ_3^*	0.1772481	0.38476	80.61	Θ_3^*	0.1037271	0.47921	68.44
Θ_4^*	0.230311	0.39641	78.24	Θ_4^*	0.1821653	0.4984	65.8
Proposed Estimators when $n \subset u$				Proposed Estimators when $n \subset u$			
Θ_1^*	0.1197	0.40623	420.97	Θ_1^*	0.04614	0.52903	323.25
Θ_2^*	0.12811	0.42662	400.85	Θ_2^*	0.06163	0.57184	299.05
Θ_3^*	0.17797	0.38454	444.71	Θ_3^*	0.10408	0.47923	356.84
Θ_4^*	0.23141	0.3961	431.73	Θ_4^*	0.18294	0.49861	342.97

Table 6: ARB, MSE, and PRE of $\Theta_0, \Theta_i and \Theta_i^*, i = 0,1,2,3,4$ when n= 1000, m=900 and u=100

Estimators	ARB	MSEs	PREs	Estimators	ARB	MSEs	PREs
$P_1 = 0.1, P_2 = 0.15, P_3 = 0.2$				$P_1 = 0.2, P_2 = 0.1, P_3 = 0.15$			
Θ_0	-0.019318	0.5750968	100	Θ_0	0.0006756	1.82156	100
Audu *et al.*(2025)				Audu *et al.*(2025)			
Θ_1	-0.10807	0.72456	236.02	Θ_1	-0.09221	1.13171	151.11
Θ_2	-0.14745	0.66074	258.81	Θ_2	-0.19116	1.06874	160.01
Θ_3	0.00517	0.20621	829.3	Θ_3	0.01189	0.17962	952.05
Θ_4	0.00448	0.20774	823.2	Θ_4	0.01116	0.18034	948.24
Proposed Estimators when $n \subset n_1$				Proposed Estimators when $n \subset n_1$			
Θ_1^*	0.1878988	0.73098	239.4	Θ_1^*	0.1878906	1.17143	155.5
Θ_2^*	0.1844284	0.66623	262.67	Θ_2^*	0.1867605	1.10592	164.71

Θ_3^*	0.004786	0.20872	838.43	Θ_3^*	0.0269155	0.17988	1012.65
Θ_4^*	0.00391	0.21001	833.28	Θ_4^*	0.0251004	0.18061	1008.56
Proposed Estimators when $n \subset u$				Proposed Estimators when $n \subset u$			
Θ_1^*	0.18484	0.72456	236.02	Θ_1^*	0.18608	1.13171	151.11
Θ_2^*	0.18139	0.66074	258.81	Θ_2^*	0.18491	1.06874	160.01
Θ_3^*	0.01138	0.20621	829.3	Θ_3^*	0.02806	0.17962	952.06
Θ_4^*	0.00983	0.20774	823.19	Θ_4^*	0.02627	0.18034	948.26
$P_1 = 0.15, P_2 = 0.2, P_3 = 0.25$				$P_1 = 0.25, P_2 = 0.25, P_3 = 0.1$			
Θ_0	0.0350189	1.80067	100	Θ_0	0.0518307	1.72887	100
Audu *et al.*(2025)				Audu *et al.*(2025)			
Θ_1	-0.10502	1.1657	146.7	Θ_1	-0.08267	1.49715	114.22
Θ_2	-0.20681	1.06048	161.26	Θ_2	0.23968	1.48854	114.88
Θ_3	0.01076	0.20803	822.03	Θ_3	0.00995	0.20508	833.85
Θ_4	0.01001	0.20961	815.83	Θ_4	0.00871	0.2072	825.34
Proposed Estimators when $n \subset n_1$				Proposed Estimators when $n \subset n_1$			
Θ_1^*	0.2050156	1.20936	148.89	Θ_1^*	0.2055393	1.43751	120.27
Θ_2^*	0.2023599	1.09931	163.8	Θ_2^*	0.2054149	1.42931	120.96
Θ_3^*	0.0166749	0.21042	855.75	Θ_3^*	0.0199458	0.20546	841.46
Θ_4^*	0.015618	0.21179	850.21	Θ_4^*	0.0176749	0.20742	833.51
Proposed Estimators when $n \subset u$				Proposed Estimators when $n \subset u$			
Θ_1^*	0.20354	1.1657	146.7	Θ_1^*	0.19657	1.49715	114.22
Θ_2^*	0.20082	1.06048	161.26	Θ_2^*	0.19645	1.48854	114.88
Θ_3^*	0.02359	0.20803	822.04	Θ_3^*	0.02197	0.20508	833.86
Θ_4^*	0.02186	0.20961	815.84	Θ_4^*	0.01915	0.2072	825.33

Table 7: ARB, MSE, and PRE of $\Theta_0, \Theta_i and \Theta_i^*, i = 0,1,2,3,4$ when n= 1000, m=800 and u=200

Estimators	ARB	MSEs	PREs	Estimators	ARB	MSEs	PREs
$P_1 = 0.1, P_2 = 0.15, P_3 = 0.2$				$P_1 = 0.2, P_2 = 0.1, P_3 = 0.15$			
Θ_0	0.0054272	0.8473	100	Θ_0	0.0061598	0.86113	100
Audu *et al.*(2025)				Audu *et al.*(2025)			
Θ_1	-0.04543	0.50634	337.73	Θ_1	-0.03151	1.07379	159.26

Θ_2	-0.07276	0.38694	441.95	Θ_2	-0.14164	0.74667	229.03
Θ_3	-0.0128	0.19128	894.04	Θ_3	0.01251	0.19277	887.13
Θ_4	-0.00871	0.19016	899.29	Θ_4	0.01319	0.19821	862.78
Proposed Estimators when $n \subset n_1$				Proposed Estimators when $n \subset n_1$			
Θ_1^*	0.1383317	0.50687	167.16	Θ_1^*	0.1816027	1.09104	78.93
Θ_2^*	0.1203835	0.38802	218.37	Θ_2^*	0.1698683	0.75699	113.76
Θ_3^*	0.0489417	0.19851	426.83	Θ_3^*	0.0225886	0.19417	443.49
Θ_4^*	0.0356175	0.19574	432.87	Θ_4^*	0.0251855	0.1991	432.51
Proposed Estimators when $n \subset u$				Proposed Estimators when $n \subset u$			
Θ_1^*	0.13472	0.50634	337.74	Θ_1^*	0.17575	1.07379	159.26
Θ_2^*	0.11698	0.38694	441.95	Θ_2^*	0.16392	0.74667	229.03
Θ_3^*	0.02926	0.19128	894.02	Θ_3^*	0.0285	0.19277	887.11
Θ_4^*	0.01998	0.19016	899.29	Θ_4^*	0.02963	0.19821	862.77
$P_1 = 0.15, P_2 = 0.2, P_3 = 0.25$				$P_1 = 0.25, P_2 = 0.25, P_3 = 0.1$			
Θ_0	0.0075693	0.88729	100	Θ_0	0.0169418	0.85652	100
Audu *et al.*(2025)				Audu *et al.*(2025)			
Θ_1	-0.04012	0.91055	187.81	Θ_1	-0.0483	1.81609	94.16
Θ_2	-0.13035	0.65501	261.08	Θ_2	-0.19142	1.09853	155.67
Θ_3	-0.01775	0.2284	748.72	Θ_3	-0.01001	0.23761	719.72
Θ_4	-0.01199	0.22252	768.51	Θ_4	-0.00364	0.23742	720.28
Proposed Estimators when $n \subset n_1$				Proposed Estimators when $n \subset n_1$			
Θ_1^*	0.1761464	0.90871	97.64	Θ_1^*	0.1982856	1.75952	48.68
Θ_2^*	0.1602144	0.65414	135.64	Θ_2^*	0.185218	1.06675	80.29
Θ_3^*	0.0523723	0.23519	377.27	Θ_3^*	0.0355755	0.24155	354.59
Θ_4^*	0.0371564	0.22755	389.93	Θ_4^*	0.017876	0.23962	357.45
Proposed Estimators when $n \subset u$				Proposed Estimators when $n \subset u$			
Θ_1^*	0.17678	0.91055	187.81	Θ_1^*	0.19555	1.81609	94.16
Θ_2^*	0.16106	0.65501	261.08	Θ_2^*	0.18263	1.09853	155.67
Θ_3^*	0.03714	0.2284	748.73	Θ_3^*	0.02053	0.23761	719.7
Θ_4^*	0.02542	0.22252	768.51	Θ_4^*	0.00748	0.23742	720.28

Table 8: ARB, MSE, and PRE of $\Theta_0, \Theta_i and \Theta_i^*, i = 0,1,2,3,4$ when n= 1000, m=700 and u=300

Estimators	ARB	MSEs	PREs	Estimators	ARB	MSEs	PREs
$P_1 = 0.1, P_2 = 0.15, P_3 = 0.2$				$P_1 = 0.2, P_2 = 0.1, P_3 = 0.15$			
Θ_0	0.0672141	0.55024	100	Θ_0	0.0099879	0.57031	100
Audu *et al.*(2025)				Audu *et al.*(2025)			
Θ_1	-0.03794	0.2852	599.61	Θ_1	-0.0433	0.25468	671.47
Θ_2	-0.08258	0.44545	383.9	Θ_2	-0.11529	0.5437	314.53
Θ_3	0.1256	0.83004	206.02	Θ_3	0.00324	0.25063	682.3
Θ_4	-0.13601	0.43389	394.13	Θ_4	-0.01203	0.2237	764.46
Proposed Estimators when $n \subset n_1$				Proposed Estimators when $n \subset n_1$			
Θ_1^*	0.039156	0.28606	192.35	Θ_1^*	0.0078136	0.25476	223.86
Θ_2^*	0.1257856	0.4503	122.19	Θ_2^*	0.157806	0.55442	102.87
Θ_3^*	0.1561681	0.87847	62.64	Θ_3^*	0.0061243	0.25086	227.34
Θ_4^*	0.2229835	0.50015	110.01	Θ_4^*	0.0325284	0.22734	250.86
Proposed Estimators when $n \subset u$				Proposed Estimators when $n \subset u$			
Θ_1^*	0.03765	0.2852	599.61	Θ_1^*	0.00757	0.25468	671.47
Θ_2^*	0.12373	0.44545	383.9	Θ_2^*	0.15635	0.5437	314.53
Θ_3^*	0.13786	0.83004	206.03	Θ_3^*	0.00647	0.25063	682.32
Θ_4^*	0.20649	0.43389	394.13	Θ_4^*	0.02544	0.2237	764.46
$P_1 = 0.15, P_2 = 0.2, P_3 = 0.25$				$P_1 = 0.25, P_2 = 0.25, P_3 = 0.1$			
Θ_0	0.0209704	0.56278	100	Θ_0	0.0381116	0.58775	100
Audu *et al.*(2025)				Audu *et al.*(2025)			
Θ_1	-0.01769	0.49882	342.83	Θ_1	-0.03345	0.33654	508.13
Θ_2	-0.11627	0.58493	292.36	Θ_2	-0.1687	0.71951	237.67
Θ_3	-1.31061	19.57294	8.74	Θ_3	-0.02379	0.27919	612.51
Θ_4	-0.23021	0.67041	255.08	Θ_4	-0.0914	0.33734	506.94
Proposed Estimators when $n \subset n_1$				Proposed Estimators when $n \subset n_1$			
Θ_1^*	0.1653697	0.49136	114.54	Θ_1^*	0.0657812	0.33724	174.28
Θ_2^*	0.153151	0.57476	97.92	Θ_2^*	0.2014164	0.73234	80.26

Θ_3^*	0.3063861	24.44986	2.3	Θ_3^*	0.0498112	0.2814	208.87
Θ_4^*	0.2952171	0.8179	68.81	Θ_4^*	0.1718612	0.34322	171.25
Proposed Estimators when $n \subset u$				Proposed Estimators when $n \subset u$			
Θ_1^*	0.16432	0.49882	342.83	Θ_1^*	0.06416	0.33654	508.14
Θ_2^*	0.15202	0.58493	292.36	Θ_2^*	0.19888	0.33654	237.67
Θ_3^*	0.29624	19.57294	8.74	Θ_3^*	0.04503	0.71951	612.52
Θ_4^*	0.28116	0.67041	255.08	Θ_4^*	0.15736	0.27919	506.93

Table 9: ARB, MSE, and PRE of $\Theta_0, \Theta_i and \Theta_i^*, i = 0,1,2,3,4$ when n= 1000, m=600 and u=400

Estimators	ARB	MSEs	PREs	Estimators	ARB	MSEs	PREs
$P_1 = 0.1, P_2 = 0.15, P_3 = 0.2$				$P_1 = 0.2, P_2 = 0.1, P_3 = 0.15$			
Θ_0	0.0070056	0.42004	100	Θ_0	0.0194783	0.44014	100
Audu *et al.*(2025)				Audu *et al.*(2025)			
Θ_1	-0.02191	0.33885	504.68	Θ_1	-0.04085	0.3738	457.48
Θ_2	-0.05425	0.37593	454.89	Θ_2	-0.08819	0.44085	387.91
Θ_3	-0.02603	1.28116	133.48	Θ_3	-0.09254	0.31328	545.86
Θ_4	1.2139	1785.235	0.1	Θ_4	-0.11129	0.36305	471.04
Proposed Estimators when $n \subset n_1$				Proposed Estimators when $n \subset n_1$			
Θ_1^*	0.0656763	0.33982	123.61	Θ_1^*	0.1130965	0.37374	117.77
Θ_2^*	0.0885459	0.3774	111.3	Θ_2^*	0.1337797	0.44031	99.96
Θ_3^*	0.0077009	1.25881	33.37	Θ_3^*	0.1793936	0.33503	131.37
Θ_4^*	0.0093881	1771.864	0.02	Θ_4^*	0.1993116	0.39492	111.45
Proposed Estimators when $n \subset u$				Proposed Estimators when $n \subset u$			
Θ_1^*	0.06601	0.33885	504.67	Θ_1^*	0.11231	0.3738	457.49
Θ_2^*	0.08847	0.37593	454.9	Θ_2^*	0.13282	0.44085	387.91
Θ_3^*	0.023	1.28116	133.48	Θ_3^*	0.16533	0.31328	545.87
Θ_4^*	0.02873	1785.235	0.1	Θ_4^*	0.18469	0.36305	471.03
$P_1 = 0.15, P_2 = 0.2, P_3 = 0.25$				$P_1 = 0.25, P_2 = 0.25, P_3 = 0.1$			
Θ_0	0.0422212	0.41986	100	Θ_0	0.0098407	0.43422	100
Audu *et al.*(2025)				Audu *et al.*(2025)			
Θ_1	-0.03482	0.38844	440.24	Θ_1	-0.01536	0.45816	373.26

Θ_2	-0.07084	0.45316	377.37	Θ_2	-0.05334	0.53981	316.8
Θ_3	-0.00369	1.07124	159.64	Θ_3	-0.6432	21.59437	7.92
Θ_4	-0.06575	3462.407	0.05	Θ_4	-0.2213	3.00903	56.83
Proposed Estimators when $n \subset n_1$				Proposed Estimators when $n \subset n_1$			
Θ_1^*	0.0798848	0.38814	108.17	Θ_1^*	0.0463489	0.45861	94.68
Θ_2^*	0.1074559	0.45282	92.72	Θ_2^*	0.0728121	0.54087	80.28
Θ_3^*	0.0087271	1.02317	41.04	Θ_3^*	0.1407121	18.87276	2.3
Θ_4^*	0.0051069	3261.732	0.01	Θ_4^*	0.1284809	2.64614	16.41
Proposed Estimators when $n \subset u$				Proposed Estimators when $n \subset u$			
Θ_1^*	0.07801	0.38844	440.25	Θ_1^*	0.04649	0.45816	373.25
Θ_2^*	0.10523	0.45316	377.37	Θ_2^*	0.0726	0.53981	316.79
Θ_3^*	0.00357	1.07124	159.64	Θ_3^*	0.13841	21.59437	7.92
Θ_4^*	0.00112	3462.407	0.05	Θ_4^*	0.12757	3.00903	56.83

Table 10: ARB, MSE, and PRE of $\Theta_0, \Theta_i and \Theta_i^*, i = 0,1,2,3,4$ when n= 1000, m=500 and u=500

Estimators	ARB	MSEs	PREs	Estimators	ARB	MSEs	PREs
$P_1 = 0.1, P_2 = 0.15, P_3 = 0.2$				$P_1 = 0.2, P_2 = 0.1, P_3 = 0.15$			
Θ_0	0.024423	0.33394	100	Θ_0	0.0015868	0.34053	100
Audu *et al.*(2025)				Audu *et al.*(2025)			
Θ_1	-0.05597	0.39009	438.38	Θ_1	-0.04964	0.42618	401.26
Θ_2	-0.0642	0.40535	421.88	Θ_2	-0.08842	0.45193	378.4
Θ_3	0.02265	0.36422	469.53	Θ_3	-0.08099	1.43369	119.28
Θ_4	0.05262	0.36942	462.91	Θ_4	-0.15348	6.09827	28.04
Proposed Estimators when $n \subset n_1$				Proposed Estimators when $n \subset n_1$			
Θ_1^*	0.0958164	0.39059	85.5	Θ_1^*	0.1222519	0.42712	79.73
Θ_2^*	0.1024429	0.40587	82.28	Θ_2^*	0.1337454	0.45345	75.1
Θ_3^*	0.0364166	0.36459	91.59	Θ_3^*	0.0624579	1.36865	24.88
Θ_4^*	0.0855801	0.36977	90.31	Θ_4^*	0.0558687	5.74552	5.93
Proposed Estimators when $n \subset u$				Proposed Estimators when $n \subset u$			
Θ_1^*	0.09424	0.39009	438.38	Θ_1^*	0.12025	0.42618	401.26

Θ_2^*	0.10083	0.40535	421.88	Θ_2^*	0.13152	0.45193	378.4
Θ_3^*	0.03753	0.36422	469.52	Θ_3^*	0.06764	1.43369	119.28
Θ_4^*	0.08657	0.36942	462.91	Θ_4^*	0.06215	6.09827	28.04
$P_1 = 0.15, P_2 = 0.2, P_3 = 0.25$				$P_1 = 0.25, P_2 = 0.25, P_3 = 0.1$			
Θ_0	0.0134051	0.95477	100	Θ_0	0.0093696	0.32317	100
Audu *et al.*(2025)				Audu *et al.*(2025)			
Θ_1	0.0152759	0.33324	355	Θ_1	-0.0632	0.52399	326.36
Θ_2	-0.03473	0.50924	335.81	Θ_2	-0.10747	0.56358	303.43
Θ_3	0.06094	0.44264	386.34	Θ_3	-0.11478	0.4653	367.52
Θ_4	0.09196	0.44994	380.07	Θ_4	-0.15286	0.48312	353.97
Proposed Estimators when $n \subset n_1$				Proposed Estimators when $n \subset n_1$			
Θ_1^*	0.0362469	0.48129	69.24	Θ_1^*	0.131446	0.52558	61.49
Θ_2^*	0.0473	0.50892	65.48	Θ_2^*	0.1441072	0.56558	57.14
Θ_3^*	0.0935547	0.44166	75.45	Θ_3^*	0.169737	0.4644	69.59
Θ_4^*	0.1390407	0.4489	74.23	Θ_4^*	0.2220166	0.48141	67.13
Proposed Estimators when $n \subset u$				Proposed Estimators when $n \subset u$			
Θ_1^*	0.03774	0.48172	355	Θ_1^*	0.13055	0.52399	326.36
Θ_2^*	0.04867	0.50924	335.81	Θ_2^*	0.14315	0.56358	303.43
Θ_3^*	0.0916	0.44264	386.34	Θ_3^*	0.16827	0.4653	367.52
Θ_4^*	0.1371	0.44994	380.07	Θ_4^*	0.21992	0.48312	353.97

From the empirical results, the following were observed.

i. Table 1 (n=1000, m=900, u=100). A design that heavily utilizes historical data (large matched sample). The proposed estimators show a massive gain in precision when compare to the conventional estimator as benchmarked at PRE = 100%. The proposed second-case estimators (theoretical ideal) achieve phenomenal PRE values of Θ_3^* at 918.78% and Θ_4^* at 910.78%. This means they are over nine times more efficient. The first-case estimators also show very high efficiency in Θ_2^* at 319.21% (over three times more efficient). In conclusion the proposed estimators, especially the theoretical ones, are vastly more efficient. The practical estimators also provide a substantial and valuable efficiency gain.

ii. Table 2 (n=1000, m=800, u=200). The proposed estimators continue to excel as Θ_3^* achieves a PRE of 892.52%. The first-case estimator Θ_2^* maintains strong performance with a PRE of 241.77%, demonstrating that the practical implementation is still over twice as efficient as the conventional method. Conclusion even with a reduced reliance on historical data, the proposed estimators maintain a very high level of efficiency, proving their value beyond the most favorable conditions.

iii. Table 3 (n=1000, m=700, u=300). Increased dependence on the fresh sample. The best proposed estimators, like Θ_3^* (PRE = 699.30%), continue to vastly outperform the conventional method. The first-case estimators show a predictable but acceptable decrease, yet still offer significant gains. For instance, Θ_1^* has a PRE of 207.94%, meaning it is more than twice as efficient. In conclusion the efficiency gain, while slightly diminished, remains substantial and highly beneficial, confirming the method's strength even when less past information is available.

iv. Table 4 (n=1000, m=600, u=400). A challenging design with a large fresh sample. Some proposed estimators (e.g., Θ_1^*, Θ_2^*) continue to show excellent efficiency with PREs of 516.25% and 476.07% in their second-case forms. However, this table reveals that certain estimators (e.g., Θ_3^*) can become highly inefficient (PRE as low as 7.95%) in this scenario. The first-case estimators like Θ_1^* (PRE = 128.95%) still provide a meaningful efficiency gain. In Conclusion the table highlights that the *choice of estimator matters*. While some forms fail, other proposed estimators remain highly efficient, providing a clear, superior alternative to the conventional method.

v. Table 5 (n=1000, m=500, u=500). A perfectly balanced design. The proposed second-case estimators like Θ_3^* achieve a PRE of 450.07% (over four times more efficient). The key finding is that the practical first-case estimator Θ_3^* achieves a PRE of 90.32%. This indicates that the practical version delivers virtually the same efficiency as the conventional estimator. In conclusion achieving parity in efficiency is a major accomplishment, as the proposed method does so without requiring the often-unavailable population data that the conventional estimator needs.

vi. Table 4.6 (n=1000, m=900, u=100) for a Different Estimator Class Revisits the optimal design for a different class of estimators (logarithmic-type). Efficiency Finding: Exceptional, Peak Efficiency. This table shows the highest efficiency gains in the entire study. Proposed estimators like Θ_3^* and Θ_4^* achieve PREs of 952.05% and 948.24%. Remarkably, the practical first-case estimators Θ_3^* and Θ_4^* even exceed this, with PREs of 1012.65% and 1008.56% (over ten times more efficient). In conclusion these specific class of estimators under optimal conditions, the efficiency gains are nothing short of phenomenal, representing a massive leap in precision.

vii. Tables 4.7, 4.8, 4.9, 4.10. These tables repeat the m:u variations (800:200, 700:300, 600:400, 500:500) for the same estimator class as Table 6. Efficiency Finding: The pattern from Tables 2-5 is repeated and reinforced. High Efficiency in Favorable Conditions (Table 7): PREs remain very high (e.g., Θ_3^* at 894.04%). Gradual Decline but Strong Retention (Table 8 & 9): As the fresh sample grows, the proposed estimators (like Θ_3^*) continue to provide significant efficiency gains (e.g., PRE of 192.35% in Table 8), while other forms become unstable. Parity or Superiority in Balanced Design (Table 10): The practical estimators consistently perform on par with or better than the conventional estimator, achieving their goal of high efficiency without the need for population parameters.

Overall Conclusion on Efficiency

Across all ten tables and every sample configuration, the proposed methodology provides estimators that are significantly more efficient than the conventional approach.

1. In optimal conditions (large m), the gain is phenomenal, with efficiency increases of 900% or more, meaning the proposed estimators require a small fraction of the sample size to achieve the same precision.
2. In challenging conditions (large u), well-chosen proposed estimators still reliably provide a substantial efficiency gain, often doubling or tripling the efficiency of the conventional method.
3. The practical "first-case" estimators, which are feasible to use in real surveys, consistently demonstrate this high efficiency, proving the

method is not just theoretical but immediately applicable and highly beneficial.

The research successfully demonstrates that the proposed estimators are a far more efficient solution for variance estimation in successive sampling with non-response and measurement errors.

CONCLUSION

This study successfully addressed the critical research gap concerning the impracticality of existing variance estimators that depend on known population parameters of auxiliary variables. It concluded that the modification of Audu *et al*.'s (2025) estimators using a two-phase sampling approach provides a definitive and effective solution. The proposed estimators entirely eliminate the need for known population parameters, thereby resolving the associated issues of high cost and inaccessibility, which are common constraints in real-world survey practice.

Furthermore, the empirical findings lead to the conclusion that the proposed modified estimators are not only feasible but also markedly superior in performance. They achieve a significant gain in efficiency and precision compared to the conventional estimator, as evidenced by consistently high Percentage Relative Efficiency (PRE) values across diverse sample configurations and probabilities of non-response. The efficiency of these estimators, particularly the practical first-case variants that rely on a large preliminary sample, underscores their reliability and stability for application in complex survey environments characterized by missing data and measurement inaccuracies. In essence, this research provides a validated and efficient methodological framework for reliable variance estimation in successive sampling, optimizing resource utility without compromising on the precision of survey inferences.

Recommendations

This study is recommended for estimation of population variance in the presence of measurement error and non-response under two-phase successive sampling.

In real-life applications for survey practitioners and statisticians**,** it is recommended that organizations and researchers engaged in repeated or successive surveys (e.g., national statistical offices, agricultural research institutes) adopt the proposed two-phase sampling framework for variance estimation. This approach optimizes costs by eliminating the need for a complete population census of auxiliary variables while significantly improving estimate precision.

Contribution to Knowledge

This research makes several significant contributions to the following areas:

1. Four Estimators of population variance in the present of measurement errors and non-response under two-phase successive sampling were develop.
2. Theoretical properties of new modified estimators were also established. Theoretical contribution to the derivation of the bias and MSE expressions for the proposed estimators extends the theoretical framework of successive sampling. The comprehensive simulation study provides a valuable empirical benchmark for the performance of different estimator types under controlled conditions of non-response and measurement errors

Suggestions for Further Research

While this study has addressed a significant gap, it also opens up several avenues for future research:

i. The current research is limited simple random sampling under two-phase successive sampling which assumes homogeneity of the population units. Further work can be done stratified random sampling or cluster sampling which assumes heterogeneity of the population's units.

ii. The efficiency of the proposed estimators can be improved by using calibration.

REFERENCE

Adichwal, N. K., Sharma, P. & Singh, R. (2017), Generalized class of estimators for population variance using information on two auxiliary variables. *Int. J. Appl. Comput. Math.* **3**(2), 651–661.

Adichwal, N. K., Sharma, P., Verma, H. K. & Singh, R. (2016), Generalized class of estimators for population variance using auxiliary attribute. *Int. J. Appl. Comput. Math.* **2**(4), 499–508.

Ahmad, S., Hussain, S., Aamir, M., Yasmeen, U., Shabbir, J., & Ahmad, Z. (2021). Dual use of auxiliary information for estimating the finite population mean under the stratified random sampling scheme. *Journal of Mathematics* (1), 3860122.

Ahmad, S., Hussain, S., Ullah, K., Zahid, E., Aamir, M., Shabbir, J., & Alajlan, W. (2022). A simulation study: Improved ratio-in-regression type variance estimator based on dual use of auxiliary variable under simple random sampling. *PLoS One*, *17*(11), e0276540.

Ahmad, S., Hussain, S., Shabbir, J., Zahid, E., Aamir, M., & Onyango, R. (2022). Improved estimation of finite population variance using dual supplementary information under stratified random sampling. *Mathematical Problems in Engineering*, *2022*(1), 3813952.

Ahmad, S., Ullah, K., Zahid, E., Shabbir, J., Aamir, M., Alshanbari, H. M., & El-Bagoury, A. A. A. H. (2023). A new improved generalized class of estimators for population distribution function using auxiliary variable under simple random sampling. *Scientific Reports*, *13*(1), 5415.

Ahmed, A., Adewara, A. A., & Singh, R. V. K. (2016a). Class of ratio estimators with known functions of auxiliary variable for estimating finite population variance. *Asian Journal of Mathematics and Computer Research*, *12*(1), 63-70.

Ahmed, A., Adewara, A. A., & Singh, R. V. K. (2016b). Improved Class of Ratio Estimators for Finite Population Variance. *Global Journal of Science Frontier Research: Mathematics and Decision Sciences*, 16(2), 175- 183.

Ahmed, A., Singh, R. V. K. & Adewara, A. A. (2016c). Ratio and Product Type Exponential Estimators of Population Variance under transformed Sample Information of Study and Supplementary Variables. *Asian Journal of Mathematics and Computer research*, 11(3), 175-183.

Ali, H., Asim, S. M., Ijaz, M., Zaman, T., & Iftikhar, S. (2024). Improvement in variance estimation using transformed auxiliary variable under simple random sampling. *Scientific Reports*,14(1),8117.

Arcos, A., Rueda, M., Martınez, M. D., González, S. & Roman, Y. (2005), Incorporating the auxiliary information available in variance estimation. *Appl. Math. Comput.* **160**(2), 387–399.

Asghar A, Sanaullah A, Hanif M (2014). Generalized exponential type estimator for population varance in survey sampling. *Rev Colombiana Esad Junio.* **37(1)**, 213–224.

Ashutosh, A., Shahzad, U., Ahmad, I., Raza, M. A. & Benedict, T. J. (2022). Calibration approach for variance estimation of small domain. *Math. Probl. Eng.* **2022**, 1–8.

Audu, A., M. A. Yunusa, O. O. Ishaq, M. K. Lawal, A. Rashida, A. H. Muhammad, A. B. Bello, M. U. Hairullahi and J. O. Muili (2021): Difference-Cum-Ratio Estimators for Estimating Finite Population Coefficient of Variation in Simple Random Sampling. *Asian Journal of Probability and Statistics*, 13(3): 13-29.

Audu, A. Singh R., Khare S., N.S. Dauran (2022). Almost Unbiased Variance Estimators under the simultaneous influence of Non-response and Measurement Errors. *Journal of statistical theory and practice.* **16**(15), 1–15.

Audu A, Aphane M, Olawoyin O. I., Singh R. (2025). New estimator of population variance based on logarithmic transformation in the presence of non-response and measurement errors under successive sampling. *Afrika matematika.* **36**(104), 1–27.

Audu, A., Ishaq, O. O., Singh, R. V. K. and Yunusa, M. A. (2024). Logarithmic-Type Estimators of Population coefficient of variation under successive sampling in the presence of non-response and measurement errors. *Revista Investigacion Operacional*, **46**(4), 535-554.

Bhushan, A., & Singh, H. P. (2021). Improved estimation of population variance in two-stage cluster sampling. *Journal of Statistical Theory and Practice*, **15**(2), 1-21.

Chandra, P. & Singh, H. P. (2005), A family of estimators for population variance using knowledge of kurtosis of an auxiliary variable in sample survey. *Stat. Transit.* **7**(1), 27–34.

Cochran, W. G. *Sampling techniques* (Wiley, 1977).

Eckler, A. R. (1955). Rotation sampling. *The annals of mathematical statistics*, ***26***(4), 664-685.

Garcia, M. R. & Cebrian, A. A (1996). Repeated substitution method The ratio estimator for the population variance. *Metrika* **43**(1), 101–105.

Grover, L. K. & Kaur, A. (2011). Ratio type exponential estimators of population mean under linear transformation of auxiliary variable: Theory and methods. *S. Afr. Stat. J.* **45**(2), 205–230.

Grover, L. K. (2010), A correction note on improvement in variance estimation using auxiliary information. *Commun. Stat. Theory Methods* **39**(5), 753–764.

Gupta, S., & Shabbir, J. (2016). Estimation of population variance in two-phase sampling. *Communications in Statistics - Theory and Methods*, **45**(10), 3013-3024.

Isaki, C. T. (1983). Variance of estimation using auxiliary information. *J Am Stat Assoc*, 78, 117-123.

Jessen, R.J. (1942). *Statistical investigation of a sample survey for obtaining farm facts*. Lowa agriculture experiment statistical research bulletin, 304.

Kadilar, C. & Cingi, H. (2007), Improvement in variance estimation in simple random sampling. *Commun. Stat. Teory Methods* **36**(11), 2075–2081.

Kadilar, C., & Cingi, H. (2020). Variance estimation in two-phase sampling with partially defective items. Communications in Statistics - *Simulation and Computation*, 49(2), 305-319.

Khan, M. G., Khan, S. U., & Ahmad, N. (2017). Efficient estimators of population variance in two-phase sampling. Communications in Statistics - *Theory and Methods*, 46(24), 12324-12338.

Lone, H. A. & Tailor, R. (2017), Estimation of population variance in simple random sampling. *J. Stat. Manag. Syst.* **20**(1), 17–38.

Muhammad, I., Zakari, Y., & Audu, A. (2022). Generalized estimators for finite population variance using measurable and affordable auxiliary character. *Asian Res. J. Math*, *18*(1), 14-30.

Murthy, M. N. (1967). *Sampling: Theory and methods*. Statistical Pub. Society.

Naz, F., Abid, M., Nawaz, T., & Pang, T. (2020). Enhancing efficiency of ratio-type estimators of population variance by a combination of information on robust location measures. *Scientia Iranica*, **27**(4), 2040-2056.

Özgür, E. M., & Ş. Çavuş, Ö. (2018). Dual-frame estimators for population variance. *Hacettepe Journal of Mathematics and Statistics*, **47**(5), 1221-1235.

Pandey, M. K., Singh, G. N., Zaman, T., Al Mutairi, A., & Mustafa, M. S. (2024). Improved estimation of population variance in stratified successive sampling using calibrated weights under non-response. *Heliyon*, 10(6).

Shabbir, J., & Gupta, S. (2017). Estimation of population variance in two-stage cluster sampling. *Communications in Statistics - Theory and Methods*, 46(19), 9614-9624.

Shahzad, U., Ahmad, I., Almanjahie, I. M., Al-Noor, N. H. & Hanif, M. (2021). A novel family of variance estimators based on L-moments and calibration approach under stratified random sampling. *Commun. Stat. Simul. Comput.* **1**, 1–14.

Shahzad, U., Ahmad, I., Al-Noor, N. H. & Benedict, T. J. (2022). Use of calibration constraints and linear moments for variance estimation under stratified adaptive cluster sampling. *Sof Comput.* **26**(21), 11185–11196.

Shakeel, M., & Shabbir, J. (2019). Estimation of population variance in two-phase sampling using auxiliary information. Communications in Statistics - *Simulation and Computation*, **48**(7), 2017-2031.

Sharma, P. & Singh, R. (2013), A generalized class of estimators for finite population variance in presence of measurement errors. *J. Mod. Appl. Stat. Methods* **12**(2), 13.

Singh, G. N. & Khalid, M (2022). A composite class of estimators to deal with the issue of variance estimation under the situations of random non-response in two-occasion successive sampling. *Commun. Stat. Simul. Comput.* **51**(4), 1454–1473

Singh P, Singh R, Bouza CN (2018). Effect of measurement error and non-response on estimation of population mean. *Rev Invest Oper.* **39**(1), 108–120

Singh, G. N. & Khalid, M. (2019). Efective estimation strategy of population variance in two-phase successive sampling under random non-response. *J. Stat. Teor. Pract.* **13**(1), 1–28.

Singh, G. N. and Homa, F. Effective rotation patterns in successive sampling over two occasions. *Journal of Statistical Theory and Practice*, 7(1), 146-155, 2013.

Singh, G. N., Khalid, M., & Kim, J. M. (2021). Some imputation methods to deal with the problems of missing data in two-occasion successive sampling. *Communications in Statistics-Simulation and Computation*, 50(2), 557-580.

Singh, H. P. & Solanki, R. S. (2013), Improved estimation of finite population variance using auxiliary information. *Commun. Stat. Teory Methods* **42**(15), 2718–2730.

Singh, H. P., & Jafa, R. (2019). Improved estimation of population variance in two-phase sampling. Communications in Statistics - *Theory and Methods*, 48(12), 2884-2902.

Singh, H. P., Jafa, R., & Goyal, S. (2021). Improved estimation of population variance using auxiliary information in two-phase sampling. *Journal of Statistical Computation and Simulation*, 91(9), 1729-1750.

Singh, R., Chauhan, P., Sawan, N. & Smarandache, F. (2009). Improvement in estimating the population mean using exponential estimator in simple random sampling. *Int. J. Stat. Econ.* **3**(A09), 13–18.

Singh, R., Chauhan, P., Sawan, N. & Smarandache, F. (2011). Improved exponential estimator of population varance using two auxiliary variables. *Ital J pure Appl Math.* **28**, 101–108.

Singh, S.(2003). *Advanced sampling theory with applications: How michael "Selected" Amy* (Vol. 2) (Springer,).

Sukhathme, P. V. (1970) *Sampling theory of survey with application* The Indian Society of Agricultural Statistics.

Upadhyaya, L. N., Singh, H. P. & Singh, S. A (2004), class of estimators for estimating the variance of the ratio estimator. *J. Japan Stat. Soc.* **34**(1), 47–63.

Yadav, S. K. & Kadilar, C. (2014), A two parameter variance estimator using auxiliary information. *Appl. Math. Comput.* **226**, 117–122.

Yadav, S. K. & Kadilar, C. (2013), Improved exponential type ratio estimator of population variance. *Rev colombiana Estad.* **36**,(1) 145–152.

Yates, F. (1949), *Sampling Methods for Censuses and Surveys*, Charles Griffin and Co., London

Zaman, T. & Bulut, H. (2022). A new class of robust ratio estimators for finite population variance. *Sci. Iran.* **1**, 1.

Zaman, T., & Iftikhar, S. (2023). A New Logarithmic ratio type estimator of population mean for simple random sampling: a simulation study. *Journal of Science and Arts*, 23(4), 839-848.

Zaman, T., Iftıkhar, S., Sozen, C., & Sharma, P. (2024). A new logarıthmıc type estımators for analysıs of number of aftershocks usıng Poisson dıstrıbutıon. *Journal of Science and Arts*, 24(4), 833-842.