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Modified Regression-Cum-Exponential Type Neutrosophic Estimators of Population Mean in the Presence of Auxiliary Information

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ABSTRACT
Neutrosophic statistics is a generalization of classical statistics and fuzzy logic that addresses ambiguous, imprecise, vague and indeterminate data. The concept of Neutrosophic statistics has been introduced in the development of estimators of population mean. The concept has extended to the development with estimators with two auxiliary variables. Some of the existing estimators of population mean with two auxiliary variables are functions of unknown parameters of auxiliary variables which makes them impracticable in real life situations. These existing estimators are either ratio-based, product-based or ratio-product-based which are less efficient when the correlation between the study and auxiliary variable is strong and negative. Therefore, to address the problem this study intends to proposed new classes of population mean estimators which are efficient and independent of unknown parameters. The proposed estimators are regression-based estimators which are applicable for either positive or negatives, weak or strong correlation. The properties (Biases and MSEs) of the proposed estimators are derived up to the first order of approximation. The theoretical efficiency conditions of the proposed estimators over some existing related estimators were established. The theoretical conclusions are validated by the empirical analysis, which made use of the simulated data. Empirical studies were conducted to assess the performances (Biases, MSEs and PREs) of the proposed estimator relative to existing estimators. The result revealed that the proposed estimators have minimum biases, minimum MSEs and higher PREs with exception of few cases compared to that of existing competing estimators considered in the study. These results demonstrate the superiority of the proposed estimator over the existing estimators in terms of the accuracy, efficiency and efficiency gain in the estimation of the population mean under Neutrosophic settings.

Keywords:
Neutrosophic data,
Estimator,
Population Mean,
Efficiency

INTRODUCTION

Sampling theory primarily aims to improve the precision of estimating unknown parameters of a population by incorporating auxiliary information. Ratio, product and regression estimators made use of auxiliary information for precision purpose. These strategies prove too efficient only when auxiliary information is available. The ratio, product and regression estimators are effective when there exists a high degree of correlation between the study variable and auxiliary variable. Among these, ratio and product estimators are widely employed in survey sampling due to their ability to yield improved estimates of the population mean when reliable auxiliary data is available.

The ratio estimator, originally introduced by Cochran (1940), operates by computing the ratio of the sample mean of the study variable to the sample mean of the auxiliary variable, thereby incorporating the known auxiliary information into the estimation process. These methods serve as effective tools in reducing bias and variance, ultimately contributing to more accurate and reliable statistical inference. Auxiliary information is information used to enhance the performance of an estimators, the auxiliary information for enhancing the performance of estimators include: Skewness of X, kurtosis of X, etc. Another method of estimating the population mean of the study variable is the product estimator,

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originally proposed by Murthy (1964), which incorporates auxiliary information by multiplying the sample means of both the study variable and the auxiliary variable. Over time, this method along with the classical ratio estimator has undergone significant refinement, for instance Bahl and Tuteja (1991), proposed exponential forms of both ratio and product estimators to improve the estimation accuracy of the population mean of the study variable using auxiliary data. Researchers such as Cochran (1940), Kadilar and Cingi (2004), Singh and Solanki (2013), Singh and Audu (2015), Audu et al. (2020), Singh et al. (2020), Audu et al. (2023), Sher et al. (2025) and Singh et al. (2024, 2025) have made notable contributions in this domain.

Neutrosophic statistics is an extension of classical statistics that addresses uncertainty, imprecision, vagueness, and indeterminacy in data. It builds upon and generalizes classical point estimation and the theoretical framework of Neutrosophic was introduced by Smarandache (2014). The presence of uncertainty or indeterminacy in data leads to the development of Neutrosophic statistics. Unlike classical statistics, which relies solely on precise (crisp) numerical data, many real -world scenarios involve uncertainty, vagueness or decision support systems, weather forecasting and economic growth evaluations situations where classical models fall short due to their inability manage indeterminate data. While fuzzy logic has been a prominent tool for addressing ambiguity and imprecision, it lacks the capacity of fully account for indeterminacy. This limitation is effectively addressed through Neutrosophic statistics which extends classical and fuzzy frameworks by incorporating determinate, indeterminate and inconsistent information. According to Smarandache (2014), Neutrosophic statistical behaves identically to classical statistics in the absence of indeterminacy, making it a robust extension rather than a replacement.

Neutrosophic approaches have been applied to various fields, such as the analysis of road trafficking accident Aslam (2020), Goodness - of -Fit testing Aslam (2020) and analysis of variance using university data Aslam (2019). In sampling estimation theory, incorporating auxiliary variables generally enhance the efficiency of estimators. Various Neutrosophic adaptations of classical estimators have been developed, particularly when the study variable shows a positive correlation with the auxiliary variable, Tahir et al. (2021).

Within the classical framework, both single and multiple auxiliary variables have been employed to construct several estimators. Among these, the ratio, product and regression estimators tend to yield comparable performance under ideal conditions. However, the regression estimator outperforms the ratio and product estimators when the regression line between the study and auxiliary variables does not pass through the origin.

In significance advancement, Singh (1967) introduced a ratio-cum-product estimator, demonstrating its superiority over traditional ratio and product estimators under certain conditions. Building on this, Perri (2005) propose an improved version by incorporating unbiased regression-base estimator of the auxiliary variable instead of the conventional mean, leading to enhanced efficiency. Later, Singh *et al*. (2009) a ratio-cum-product exponential ratio and regression-type estimators in terms of efficiency.

This research focuses on the application of Neutrosophic logic within the field of statistics, specially addressing Neutrosophic sets and logic. Neutrosophic methods are utilized in cases where traditional fuzzy or in functionistic statistical tools fall short in effectively handling indeterminate or imprecise data. Tahir et al. (2021) introduced Neutrosophic ratio-type estimators for estimations under certainty. In the works of Singh et al. (2025) investigates almost unbiased estimator for the population mean making use of the Neutrosophic auxiliary information. Furthermore Yadav et al. (2025) proposed almost unbiased estimators of population mean making use of the two Neutrosophic auxiliary variables information. However, the following gaps were identified, the proposed mean estimators are function of unknown parameters of auxiliary variables which makes them impracticable in real life situations and the proposed estimators are either ratio-based, product-base or ratio-product bases which are less efficient when the correlation between the study and auxiliary variable is strong and negative. To address the above limitations, new classes of population mean estimators which are efficient and independent of unknown parameters will be proposed and the proposed mean estimators are regression base estimators which are applicable for either negative or positive correlation.

Reviews on Neutrosophic Data and Terminology

Various forms of Neutrosophic observations have been introduced including quantitative Neutrosophic data, which indicate that a particular value may lie within an uncertain interval (a,b) (Smarandache, 2014). There exist multiple approaches to represent such interval valued, Neutrosophic numbers. For instance, references Tahir et al. (2021) and Yadav and Smarandache (2023) define Neutrosophic interval values as $Z_n = Z_L + Z_U I_N$, Where, $I_N \in \left(I_L, I_U\right)$. Here ‘ Z_L ’ and ‘ Z_U ’ represent the lower and upper bound of the Neutrosophic variable ‘ Z_N ’, the term ‘ I_N ’ indicates the degree of indeterminacy in ‘ Z_N ’ taking values 0 to 1. Following the notation used in Yadav (2023), also express the Neutrosophic data interval form as $Z_N \in \left(Z_L, Z_U\right)$

Consider a finite Neutrosophic population $U_N = U_{1N}, U_{2N,} U_{3N} .... U_{NN}$ consisting N_N units. Each unit in this population is uniquely identified by a label ranging from 1 to N, and all unit labels are assumed to be known.

Let $y_N \in (y_L, y_U)$ denoted the Neutrosophic study variable and let $x_{1N} \in (x_L, x_U)$ and $x_{2N} \in (x_{2L}, x_{2U})$ represent two auxiliary variables. The observed values for each units U_{iN} are given by y_{iN}, x_{iN} and x_{2iN}.

$\bar{y}_N \in (\bar{y}_L, \bar{y}_U)$, $\bar{x}_{1N} \in (\bar{x}_{1L}, \bar{x}_{2U})$ *and* $\bar{x}_{2N} \in (\bar{x}_{2L}, \bar{x}_{2U})$

are the Neutrosophic sample means corresponding to i^{th} sample observation on study variable $Y_N \in (Y_L, Y_U)$ and auxiliary variables

$X_{1N} \in (X_{1L}, X_{2U})$ *and* $X_{2N} \in (X_{2L}, X_{2U})$.

In addition the Neutrosophic coefficient of variation are given as

$C_{yN} \in (C_{yL}, C_{yU}), C_{x1N} \in (C_{x1L}, C_{x1U})$ *and* $C_{x2N} \in (C_{x2L}, C_{x2U})$. And the Neutrosophic correlation coefficients between the study and auxiliary variables are $\rho_{yx1N} \in (\rho_{yx1L}, \rho_{yx2U})$ *and* $\rho_{yx2N} \in (\rho_{yx2L}, \rho_{yx2U})$

Review of Some Existing Estimator of Neutrosophic population Mean Base on Single Auxiliary Variable

Assuming that x_N is positively correlated with y_N. The objective is to estimate the finite population mean of the study variable Y_N, define as $\bar{Y}_N = \frac{1}{N_N}\sum_{i=1}^{N} Y_{iN}$.

The usual sample mean estimator of a Neutrosophic variant as proposed by Kumar and Smarandache (2023) and the variance of the estimator is given as in (1.1) and (1.2) respectively

$$t_{0N} = \bar{y}_N, \in (t_{0L}, t_{0U}) \quad (1.1)$$

$$V(t_{0N}) = \theta_N \bar{Y}_N^2 C_{yN}^2 \quad (1.2)$$

Motivated by Cochran (1940), Tahir *et al.* (2021) proposed ratio estimator base on Neutrosophic data denoted by t_{1N} as in (1.3). The bias and MSE of t_{1N} are given in (1.4) and (1.5) respectively.

$$t_{1N} = \frac{\bar{y}_N}{x_{1N}} \bar{X}_{1N} \quad (1.3)$$

$$Bias(t_{1N}) = \theta_N \bar{Y}_N \left(C_{x1N}^2 - C_{x1N} C_{yN} \rho_{yxN}\right) \quad (1.4)$$

$$MSE(t_{1N}) = \theta_N \bar{Y}_N^2 \left(C_{yN}^2 + C_{xN}^2 - 2C_{x1N} C_{yN} \rho_{yx1N}\right) \quad (1.5)$$

Motivated by Bahl and Tuteja (1991), Tahir *et al*. (2021) proposed exponential-type ratio estimator base on Neutrosophic data denoted by t_{2N} as in (`1.6). The bias and MSE of t_{2N} are given in (1.7) and (1.8) respectively.

$$t_{2N} = \bar{y}_N \exp\left(\frac{\bar{X}_{1N} - \bar{x}_{1N}}{\bar{X}_{1N} + \bar{x}_{1N}}\right) \quad (1.6)$$

$$Bias(t_{2N}) = \theta_N \bar{Y}_N \left(\frac{3}{8} C_{x1N}^2 - \frac{1}{2} C_{xN} \rho_{yx1N}\right) \quad (1.7)$$

$$MSE(t_{2N}) = \theta_N \bar{Y}_N^2 \left(C_{yN}^2 + \frac{1}{4} C_{x1N}^2 - C_{x1N} C_{yN} \rho_{yx1N}\right) \quad (1.8)$$

Inspired by Pandey and Dubey (1988) who proposed ratio estimator with additive transformation base on Coefficient of variation, Tahir *et al* (2021) proposed exponential-type ratio estimator base on Neutrosophic data denoted by t_{3N} as in (1.9). The bias and MSE of t_{3N} are given in (1.10) and (1.11) respectively.

$$t_{3N} = \bar{y}_N \left(\frac{\bar{X}_{1N} + C_{x1N}}{\bar{x}_{1N} + C_{x1N}}\right) \quad (1.9)$$

$$Bias(t_{3N}) = \theta_N \bar{Y}_N \left[\left(\frac{\bar{X}_{1N}}{\bar{X}_{1N} + C_{xN}}\right)^2 C_{x1N} - \left(\frac{\bar{X}_{1N}}{\bar{X}_{1N} + C_{xN}}\right) C_{x1N} C_{yN} \rho_{yx1N}\right] \quad (1.10)$$

$$MSE(t_{3N}) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + \left(\frac{\bar{X}_{1N}}{\bar{X}_{1N} + C_{xN}}\right)^2 C_{x1N}^2 - 2\left(\frac{\bar{X}_{1N}}{\bar{X}_{1N} + C_{x1N}}\right) C_{x1N} C_{yN} \rho_{yx1N}\right] \quad (1.11)$$

Motivated by Singh and Tailor (2005), whom proposed the ratio estimator with additive transformation base on correlation coefficient, Kumar and Smarandache (2023) proposed Correlation coefficient base on Neutrosophic data denoted by t_{4N} as in (1.12). The bias and MSE of t_{4N} are given as in (1.13) and (1.14) respectively

$$t_{4N} = \bar{y}_N \left(\frac{\bar{X}_{1N} + \rho_{yx1N}}{\bar{x}_{1N} + \rho_{yx1N}} \right) \quad (1.12)$$

$$Bias(t_{4N}) = \theta_N \bar{Y}_N \left[\left(\frac{\bar{X}_{1N}}{\bar{X}_{1N} + \rho_{yx1N}} \right)^2 C_{x1N}^2 - \left(\frac{\bar{X}_{1N}}{\bar{X}_{1N} + \rho_{yx1N}} \right) C_{x1N} C_{yN} \rho_{yx1N} \right] \quad (1.13)$$

$$MSE(t_{4N}) = \theta_N \bar{Y}_N^2 \left[\left(\frac{\bar{X}_{1N}}{\bar{X}_{1N} + \rho_{yx1N}} \right)^2 C_{yN}^2 + C_{x1N}^2 - 2 \left(\frac{\bar{X}_{1N}}{\bar{X}_{1N} + \rho_{yx1N}} \right) C_{x1N} C_{yN} \rho_{yx1N} \right] \quad (1.14)$$

Inspired by Bahl and Tuteja (1991), who proposed ratio type exponential, and Tahir et al. (2021) proposed exponential type ratio estimator base on Neutrosophic data denoted by t_{5N} as in (1.15)

$$t_{5N} = \bar{y}_N \exp\left(\frac{\bar{x}_{2N} - \bar{X}_{2N}}{\bar{x}_{2N} - \bar{X}_{2N}} \right) \quad (1.15)$$

$$Bias(t_{5N}) = \theta_N \bar{Y}_N \left(\frac{3}{8} C_{xN}^2 - \frac{1}{2} C_{yxN} \right) \quad (1.16)$$

$$MSE(t_{5N}) = \theta_N \bar{Y}_N^2 \left(C_{yN}^2 + \frac{C_{xN}^2}{4} - C_{yxN} \right) \quad (1.17)$$

Motivated by Upadlyaya and Singh (1999), Yadav et al. (2025), proposed the Neutrosophic ratio estimators utilizing coefficient of variation and the coefficient of kurtosis of auxiliary variable X, express as t_{7N} given in (1.18). The bias and MSE are given below:

$$t_{6N} = \bar{y}_N \left(\frac{\beta_{2(x)N} \bar{X}_N + C_{xN}}{\beta_{2(x)N} \bar{x}_N + C_{xN}} \right) \quad (1.18)$$

$$Bias(t_{6N}) = \theta_N \bar{Y}_N \left(\lambda_{2N}^2 C_{xN}^2 - \lambda_{2N} C_{yxN} \right) \quad (1.19)$$

$$MSE(t_{6N}) = \theta_N \bar{Y}_N^2 \left(C_{yN}^2 + \lambda_{2N}^2 C_{xN}^2 - 2\lambda_{2N} C_{yxN} \right) \quad (1.20)$$

Review of Some Existing Estimators of Neutrosophic Population Mean Base on Two Auxiliary Variables

Inspired by the work of Singh (1969), Yadav *et al.* (2025) proposed an improved upon the traditional ratio-cum-product estimator, denoted as t_{RPN} in (1.21) for estimating the finite population mean under a Neutrosophic framework. The bias MSE are given as (1.22) and (1.23) respectively

$$t_{RPN} = \bar{y}_N \left(\frac{\bar{X}_{1N}}{\bar{x}_{1N}} \right) \left(\frac{\bar{x}_{2N}}{\bar{X}_{2N}} \right) \quad (1.21)$$

$$Bias(t_{RPN}) = \theta_N \bar{Y}_N \left(C_{x1N}^2 - \rho_{yx1N} C_{yN} C_{x1N} - \rho_{yx2N} C_{yN} C_{x2N} + \rho_{x1x2N} C_{x1N} C_{2N} \right) \quad (1.22)$$

$$MSE(t_{RPN}) = \theta_N \bar{Y}_N^2 \left(C_{yN}^2 + C_{x1N}^2 + C_{x2N}^2 - 2\rho_{yx1N} C_{yN} C_{x1N} + 2\rho_{yx2N} C_{yN} C_{x2N} - 2\rho_{x1x2N} C_{x1N} C_{x2N} \right) \quad (1.23)$$

Motivated by Singh (1969), Yadav *et al*. (2025) proposed product-cum-product estimator as denoted as t_{PPN} in (1.24). The bias and MSE are represents as (1.25) and (1.26)

$$t_{PPN} = \bar{y}_N \left(\frac{\bar{x}_{1N}}{\bar{X}_{1N}} \right) \left(\frac{\bar{x}_{2N}}{\bar{X}_{2N}} \right) \quad (1.24)$$

$$Bias(t_{PPN}) = \theta_N \bar{Y}_N \left(\rho_{yx1N} C_{yN} C_{x1N} + \rho_{yx2N} C_{yN} C_{x2N} + \rho_{x1x2N} C_{x1N} C_{x2N} \right) \quad (1.25)$$

$$MSE(t_{PPN}) = \theta_N \bar{Y}_N^2 \left(C_{yN}^2 + C_{x1N}^2 + C_{x2N}^2 + 2\rho_{yx1N} C_{yN} C_{x1N} + 2\rho_{yx2N} C_{yN} C_{x2N} + 2\rho_{x1x2N} C_{x1N} C_{2N} \right) \quad (1.26)$$

Yadav *et al* (2025) proposed almost unbiased estimators. The bias and MSE of t_{hN}, are as in (1.27) and (1.30) respectively. The biases and MSEs are presented in (1.28), (1.29), (1.31) and (1.32) respectively.

$$t_{hN} = \alpha_{0N}\bar{y}_N + \alpha_{1N}\bar{y}_N\left(\frac{\bar{X}_{1N}}{\bar{x}_{1N}}\right)\left(\frac{\bar{X}_{2N}}{\bar{x}_{2N}}\right) + \alpha_{2N}\bar{y}_N\left(\frac{\bar{x}_{1N}}{\bar{X}_{1N}}\right)\left(\frac{\bar{x}_{2N}}{\bar{X}_{2N}}\right) \quad (1.27)$$

$$Bias\left(t_{hN}\right) = \theta_N\bar{Y}_N\begin{bmatrix}\alpha_{1N}\left(C_{x1N}^2 + C_{x2N}^2\right) + \\ H_N\rho_{yx1N}C_{yN}C_{x1N} + \\ H_N\rho_{yx2N}C_{yN}C_{x1N} \\ +\left(\alpha_{1N}+\alpha_{2N}\right)\rho_{x1x2N}C_{x1N}C_{x2N}\end{bmatrix} \quad (1.28)$$

$$MSE\left(t_{hN}\right) = \theta_N\bar{Y}_N^2\begin{pmatrix}C_{yN}^2 + H_N^2C_{x1N}^2 + H_N^2C_{x2N}^2 \\ +2H_N\rho_{yx1N}C_{yN}C_{x1N} \\ +2H_N\rho_{yx2N}C_{yN}C_{x2N} + \\ 2H_N^2\rho_{x1x2N}C_{x1N}C_{x2N}\end{pmatrix} \quad (1.29)$$

$$t_{hIN} = I_{0N}\bar{y}_N + I_{1N}\bar{y}_N\left(\frac{\bar{X}_{1N}}{\bar{x}_{1N}}\right)\left(\frac{\bar{x}_{2N}}{\bar{X}_{2N}}\right) + I_{2N}\bar{y}_N\exp\left(\frac{\bar{X}_{1N}-\bar{x}_{1N}}{\bar{X}_{1N}+\bar{x}_{1N}}\right)\exp\left(\frac{\bar{x}_{2N}-\bar{X}_{2N}}{\bar{x}_{2N}+\bar{X}_{2N}}\right) \quad (1.30)$$

$$Bias\left(t_{hIN}\right) = \theta_N\bar{Y}_N\begin{bmatrix}H_{IN}C_{x1N}^2 + H_{IN}\rho_{yx2N}C_{yN}C_{x2N} \\ +\frac{1}{4}I_{2N}\rho_{x1x2N}C_{x1N}C_{x2N} \\ -H_{IN}\rho_{yx1N}C_{yN}C_{x1N} - \\ H_{IN}\rho_{x1x2N}C_{x1N}C_{x2N} - \frac{1}{8}I_{2N}C_{x1N}^2 - \\ \frac{1}{4}I_{2N}C_{x2N}^2\end{bmatrix} \quad (1.31)$$

$$MSE\left(t_{hIN}\right) = \theta_N\bar{Y}_N^2\begin{pmatrix}C_{yN}^2 + H_{IN}^2C_{x1N}^2 + C_{x2N}^2 \\ -2H_{IN}\rho_{yx1N}C_{yN}C_{x1N} \\ +2H_{IN}\rho_{yx2N}C_{yN}C_{x1N} \\ -2H_{IN}^2\rho_{x1x2N}C_{x1N}C_{x2N}\end{pmatrix} \quad (1.32)$$

Several estimators have been suggested by Yadav et al. (2025) and are shown to be efficient. However, their population means estimators depend on unknown constants which make it impracticable in real life situations; also their estimator is ratio-base-estimator which is less efficient when the correlation between the study and auxiliary variable is negative. To address the above gaps in Yadav el al. (2025) prompted the current study.

MATERIALS AND METHODS

Proposed Population Mean Estimators under Neutrosophic Setting

Having studied the work of Yadav *et al*. (2025) and identified the various limitations or gaps in their study, we proposed the following estimators as in (1.33), (1.34), (1.35), (1.36) and (1.37)

$$T_{p1j} = \left[\bar{y}_N + b_{N1}\left(\bar{X}_{1N}-\bar{x}_{1N}\right) + b_{N2}\left(\bar{X}_{2N}-\bar{x}_{2N}\right)\right] \times \exp\left(\frac{\left(K_i\bar{X}_{1N}+l_i\right)-\left(K_i\bar{x}_{1N}+l_i\right)}{\left(K_i\bar{X}_{1N}+l_i\right)+\left(K_i\bar{x}_{1N}+l_i\right)}\right) \times \exp\left(\frac{\left(K_i\bar{X}_{2N}+l_i\right)-\left(K_i\bar{x}_{2N}+l_i\right)}{\left(K_i\bar{X}_{2N}+l_i\right)+\left(K_i\bar{x}_{2N}+l_i\right)}\right)$$

(1.33)

$$T_{p2j} = \begin{bmatrix}\bar{y}_N + b_{1N}\left(\bar{X}_{1N}-\bar{x}_{1N}\right) + \\ b_{2N}\left(\bar{X}_{2N}-\bar{x}_{2N}\right)\end{bmatrix} \times \left(\frac{K_i\bar{X}_{1N}+l_i}{K_i\bar{x}_{1N}^*+l_i}\right) \exp\left(\frac{\left(K_i\bar{x}_{2N}^*+l_i\right)-\left(K_i\bar{X}_{2N}+l_i\right)}{\left(K_i\bar{x}_{2N}^*+l_i\right)+\left(K_i\bar{X}_{2N}+l_i\right)}\right) \quad (1.34)$$

$$T_{P3j} = '$$

$$\left[\begin{bmatrix}\bar{y}_N + b_{1N}\left(\bar{X}_{1N} - \bar{x}_{1N}\right) + \\ b_{2N}\left(\bar{X}_{2N} - \bar{x}_{2N}\right)\end{bmatrix}\left(\frac{K_i\bar{X}_{1N} + l_i}{K_i\bar{x}_{1N} + l_i}\right)\times \atop \exp\left(\frac{\left(K_i\bar{X}_{2i} + l_i\right) - \left(K_i\bar{x}_{2N} + l_i\right)}{\left(K_i\bar{X}_{2N} + l_i\right) + \left(K_i\bar{x}_{2N} + l_i\right)}\right)\right] \quad (1.35)$$

$$T_{p4j} =$$

$$\left[\bar{y}_N + b_{1N}\left(\bar{X}_{1N} - \bar{x}_{1N}\right) + b_{2N}\left(\bar{X}_{2N} - \bar{x}_{2N}\right)\right] \quad (1.36)$$

$$\left(\frac{K_i\bar{X}_{iN} + l_i}{K_i\bar{x}_{1N} + l_i}\right)\left(\frac{K_i\bar{X}_{2N} + l_i}{K_i\bar{x}_{2N} + l_i}\right)$$

$$T_{P5j} =$$

$$\begin{bmatrix}\bar{y}_N + b_{1N}\left(\bar{X}_{1N} - \bar{x}_{1N}\right) \\ + b_2\left(\bar{X}_{2N} - \bar{x}_{2N}\right)\end{bmatrix}\left(\frac{K_i\bar{X}_{1N} + l_i}{K_i\bar{x}_{1N} + l_i}\right) \quad (1.37)$$

$$\left(\frac{K_i\bar{X}_{2N} + l_i}{K_i\bar{x}_{2N} + l_i}\right)$$

Where $\bar{x}_{1N}^* = \frac{N\bar{X}_{1N} - n\bar{x}_{1n}}{N-n}$, $\bar{x}_{2N}^* = \frac{N\bar{X}_{2N} - n\bar{x}_{2N}}{N-n}$,

$b_{1N} = \frac{\rho_{yx_1N}S_{yN}}{S_{X1N}}$, $b_{2N} = \frac{\rho_{yx_2N}S_{yN}}{S_{x_2N}}$,

Procedure for Deriving the Properties (Biases and MSEs) of the Proposed New Estimator

In this sub section, the procedures for obtaining the properties (Biases and MSEs) of the proposed estimators were presented. However, to obtain the Biases and MSEs of the proposed estimators, the following error terms are defined

$$\begin{aligned} &E\left(e_{0N}\right) = E\left(e_{1N}\right) = E\left(e_{2N}\right) = 0, \\ &E\left(e_{0N}^2\right) = \gamma_N C_{yN}^2, E\left(e_{1N}^2\right) = \gamma_N C_{x1N}^2, \\ &E\left(e_{2N}^2\right) = \gamma_N C_{x2N}^2 \end{aligned} \quad (1.38)$$

$$Bias\left(T_{pj}\right) = E\left(T_{pj} - \bar{Y}_N\right) \quad \text{Where, } j = 1,2,3,4,5 \quad (1.39)$$

$$MSE\left(T_{pj}\right) = E\left(T_{pj} - \bar{Y}_N\right)^2 \quad \text{where, } j = 1,2,3,4,5 \quad (1.40)$$

The Bias and Mean Square Error (MSE) of estimators t_1, t_2, t_3, t_4, t_5 are obtained by using the results of the expected value of error terms in (1.38)

$E\left(e_{0N}^2\right) = \gamma_N C_{yN}^2$, $E\left(e_{1N}^2\right) = \gamma_N C_{x1N}^2$,
$E\left(e_{2N}^2\right) = \gamma_N C_{x2N}^2$, $E\left(e_{0N}e_{1N}\right) = \gamma_N C_{yxN}$,
$E\left(e_{0N}e_{2N}\right) = \gamma_N C_{yx2N}$, $E\left(e_{1N}e_{2N}\right) = \gamma_N C_{x1x2N}$

PROPERTIES OF THE PROPOSED ESTIMATOR

In this section, the biases and mean square errors (MSEs) of proposed estimators were derived and presented.

Bias and MSE of the Proposed Estimator T_{P1j}

Express T_{P1j} in terms of e_{0N}, e_{1N}, e_{2N}, equation (2.1) is obtained.

$$T_{P1j} = \left[\begin{matrix}\left(1 + e_{0N}\right)\bar{Y}_N + b_{1N} \\ \left(\begin{matrix}\bar{X}_{1N} - \bar{X}_{1N}\left(1 + e_{1N}\right) + \\ b_{2N}\left(\bar{X}_{2N} - \bar{X}_{2N}\left(1 + e_{2N}\right)\right)\end{matrix}\right)\end{matrix}\right]$$

$$\times \exp\left(\frac{\left(k_{1i}\bar{X}_{1N} + l_{1i}\right) - \left(k_{1i}\bar{X}_{1N}\left(1 + e_{1N}\right) + l_{1i}\right)}{\left(k_{1i}\bar{X}_{1N} + l_{1i}\right) + \left(k_{1i}\bar{X}_{1N}\left(1 + e_{1N}\right) + l_{1i}\right)}\right)$$

$$\times \exp\left(\frac{\left(k_{2i}\bar{X}_{2N} + l_{2i}\right) - \left(k_{2i}\bar{X}_{2N}\left(1 + e_{2N}\right) + l_{2i}\right)}{\left(k_{2i}\bar{X}_{2N} + l_{2i}\right) + \left(k_{2i}\bar{X}_{2N}\left(1 + e_{2N}\right) + l_{2i}\right)}\right) \quad (2.1)$$

Simplifying equation (4.2) to first order approximation we have;

$$\begin{aligned} T_{P1j} &= \left(\begin{matrix}\bar{Y}_N + \bar{Y}_N e_{0N} - \\ b_{1N}\bar{X}_{1N}e_{1N} - b_{2N}\bar{X}_{2N}e_{2N}\end{matrix}\right) \\ &\exp\left(\frac{-\theta_{1i}}{2}e_{1N} + \frac{\theta_{1i}^2}{4}\right)\times \\ &\exp\left(\frac{-\theta_{2i}}{2}e_{2N} + \frac{\theta_{2i}^2}{4}e_{2N}^2\right) \end{aligned} \quad (2.2)$$

where $\theta_{1i} = \frac{k_{1i}\bar{X}_{1N}}{k_{1I}\bar{X}_{1N} + l_{1i}}$, $\theta_{2i} = \frac{k_{2i}\bar{X}_{2N}}{k_{2i}\bar{X}_{2N} + l_{2i}}$

Take Expectation of both sides of (2.2), the Bias of the proposed estimator T_{P1j} is obtained as in equation (2.3).

$$Bias\left(T_{P1j}\right)=$$
$$\gamma_N\begin{pmatrix}\frac{-\theta_{1i}}{2}\bar{Y}_N C_{yx1N}-\frac{\theta_{2i}}{2}\bar{Y}_N C_{yx2N}\\+\left(\frac{3\theta_{1i}^2\bar{Y}_N}{8}-\frac{\theta_{1i}}{2}b_{1N}\bar{X}_{1N}\right)C_{x1N}^2\\+\left(\frac{3}{8}\theta_{2i}^2\bar{Y}_N-\frac{\theta_{2N}}{2}b_{2N}\bar{X}_{2N}\right)C_{x2N}^2\\+\left(\frac{\theta_{2i}\theta_{1i}\bar{Y}_N}{2}+\frac{\theta_{2i}b_{1N}\bar{X}_{1N}}{2}+\frac{\theta_{1i}b_{2N}\bar{X}_{2N}}{2}\right)C_{x1x2N}\end{pmatrix} \quad (2.3)$$

Square both sides of (2.3) and take Expectation of the result, the MSE of the proposed estimator T_{P1j} is obtained as in equation (2.4).

$$MSE\left(T_{P1j}\right)=$$
$$\gamma_N\begin{pmatrix}\bar{Y}_N^2C_{yN}^2+\left(\frac{\theta_{1i}}{2}\bar{Y}_N+b_{1N}\bar{X}_{1N}\right)^2C_{x1N}^2\\+\left(\frac{\theta_{2i}\bar{Y}_N}{2}+b_{1N}\bar{X}_{1N}\right)^2C_{x2N}^2\\-2\bar{Y}_N\left(\frac{\theta_{1i}\bar{Y}_N}{2}+b_{1N}\bar{X}_{1N}\right)C_{yx1N}\\-2\bar{Y}_N\left(\frac{\theta_{2i}\bar{Y}_N}{2}+b_{2N}\bar{X}_{2N}\right)C_{yx2N}\\+2\left(\frac{\theta_{1i}\bar{Y}_N}{2}+b_{1N}\bar{X}_{1N}\right)\left(\frac{\theta_{2N}\bar{Y}_N}{2}+b_{2N}\bar{X}_{2N}\right)\\(C_{x1x2N})\end{pmatrix} \quad (2.4)$$

Bias and MSE of the Proposed Estimator T_{P2j}

Express T_{P2j} in terms of e_{0N}, e_{1N}, e_{2N}, equation (2.5) is obtained.

$$T_{P2j}=\begin{pmatrix}\bar{Y}_N+\bar{Y}_N e_{oN}-b_{1N}\bar{X}_{1N}e_{1N}\\-b_{2N}\bar{X}_{2N}e_{2N}\end{pmatrix}\times\left(\frac{k_{1i}\bar{X}_{1N}+l_{1i}}{k_{1i}\bar{X}_{1N}\left(1-\lambda_{1N}e_{1N}\right)+l_{1i}}\right)\times\exp\left(\frac{\begin{matrix}\left(k_{2i}\bar{X}_{2N}\left(1-\lambda_{2N}e_{2N}\right)+l_{2i}\right)\\-\left(k_{2i}\bar{X}_{2N}+l_{2i}\right)\end{matrix}}{\begin{matrix}\left(k_{2i}\bar{X}_{2N}\left(1-\lambda_{2N}e_{2N}\right)+l_{2i}\right)\\+\left(k_{2i}\bar{X}_{2N}+l_{2i}\right)\end{matrix}}\right) \quad (2.5)$$

$$T_{P2j}=\begin{pmatrix}\bar{Y}_N+\bar{Y}_N e_{0N}-b_{1N}\bar{X}_{1N}e_{1N}\\-b_{2N}\bar{X}_{2N}e_{2N}\end{pmatrix}\times\left(\frac{1}{1-\theta_{1i}^*e_{1N}}\right)\exp\left(\frac{-\theta_{2i}^*e_{2N}}{2}\left(1-\frac{\theta_{2i}^*e_{2N}}{2}\right)^{-1}\right) \quad (2.6)$$

where $\theta_{1i}^*=\frac{k_{1i}\bar{X}_{1N}\lambda_{1N}}{k_{1i}\bar{X}_{1N}+l_{1i}},\quad \theta_{2i}^*=\frac{k_{2i}\bar{X}_{2N}\lambda_{2N}}{k_{2i}\bar{X}_{2N}+l_{2i}}$

Take Expectation of both sides of (2.6), the Bias of the proposed estimator T_{P2j} is obtained as in equation (2.7).

$$Bias\left(T_{P2j}\right)=$$
$$\gamma_N\begin{pmatrix}\theta_{1i}^*\bar{Y}_N C_{yx1N}-\frac{\theta_{2i}^*\bar{Y}_N C_{yx2N}}{2}+\\\left(\theta_{1i}^{*2}\bar{Y}_N-\theta_{1i}^*b_{1N}\bar{X}_{1N}\right)C_{x1N}^2\\+\left(\frac{3\theta_{2i}^{*2}\bar{Y}_N}{8}+\frac{\theta_{2i}^*b_{2N}\bar{X}_{2N}}{2}\right)C_{x2N}^2\\-\left(\frac{\theta_{1i}^*\theta_{2i}^*\bar{Y}_N}{2}-\frac{\theta_{2i}^*b_{1N}\bar{X}_{1N}}{2}+\theta_{1i}^*b_{2N}\bar{X}_{2N}\right)\\(C_{x1x2N})\end{pmatrix} \quad (2.7)$$

Square both sides of (2.7) and take Expectation of the result, the MSE of the proposed estimator T_{P1j} is obtained as in equation (2.8).

$$MSE\left(T_{P2j}\right)=\gamma_N\left(\begin{array}{l}\bar{Y}_N^2C_{yN}^2+\left(\theta_{1i}^*\bar{Y}_N-b_{1N}\bar{X}_{1N}\right)^2C_{x1N}^2\\+\left(\theta_{2i}^*\bar{Y}_N+b_{2N}\bar{X}_{2N}\right)^2C_{x2N}^2\\+2\bar{Y}_N\left(\theta_{1i}^2\bar{Y}_N-b_{1N}\bar{X}_{1N}\right)C_{yx1N}\\-2\bar{Y}_N\left(\frac{\theta_{2i}^*\bar{Y}_N}{2}+b_{2N}\bar{X}_{2N}\right)C_{yx2N}\\-2\left(\theta_{1i}^*\bar{Y}_N-b_{1N}\bar{X}_{1N}\right)\\\left(\frac{\theta_{2i}^*\bar{Y}_N}{2}+b_{2N}\bar{X}_N\right)C_{x1x2N}\end{array}\right) \quad (2.8)$$

Bias and MSE of the Proposed Estimator T_{P3j}

Express T_{P3j} in terms of e_{0N}, e_{1N}, e_{2N}, equation (2.9) is obtained.

$$T_{P3j}=\left[\begin{array}{l}\left(1+e_{0N}\right)\bar{Y}_N+b_{1N}\\\left(\begin{array}{l}\bar{X}_{1N}-\bar{X}_{1N}\left(1+e_{1N}\right)+\\b_{2N}\left(\bar{X}_{2N}-\bar{X}_{2N}\left(1+e_{2N}\right)\right)\end{array}\right)\end{array}\right]\times\left(\frac{k_{1i}\bar{X}_{1N}+l_{1i}}{k_{1i}\bar{X}_{1N}\left(1+e_{1N}\right)+l_{1i}}\right)\times\exp\left(\frac{\left(k_{1i}\bar{X}_{1N}+l_{1i}\right)-\left(k_{1i}\bar{X}_{1N}\left(1+e_{1N}\right)+l_{1i}\right)}{\left(k_{1i}\bar{X}_{1N}+l_{1i}\right)+\left(k_{1i}\bar{X}_{1N}\left(1+e_{1N}\right)+l_{1i}\right)}\right) \quad (2.9)$$

Simplifying equation (2.9)

$$T_{P3j}=\left(\bar{Y}_N+\bar{Y}_Ne_{0N}-b_{1N}\bar{X}_{1N}e_{1N}-b_{2N}\bar{X}e_{2N}\right)\times\left(\frac{1}{1+\theta_{1i}e_{1N}}\right)\exp\left(\frac{-\theta_{2i}e_{2N}}{2}\left(1+\frac{\theta_{2i}}{2}e_{2N}\right)^{-1}\right) \quad (2.10)$$

where $\theta_{1i}=\frac{k_{1i}\bar{X}_{1N}}{k_{1I}\bar{X}_{1N}+l_{1i}}, \quad \theta_{2i}=\frac{k_{2i}\bar{X}_{2N}}{k_{2i}\bar{X}_{2N}+l_{2i}}$

Take Expectation of both sides of (2.10), the Bias of the proposed estimator T_{P3j} is obtained as in equation (2.11).

$$Bias=\gamma_N\left(\begin{array}{l}-\theta_{1i}\bar{Y}_NC_{yx1N}-\frac{\theta_{2i}\bar{Y}_NC_{yx1N}}{2}+\\\left(\theta_{1i}^2\bar{Y}_N+\theta_{1i}b_{1N}\bar{X}_{1N}\right)C_{x1N}^2\\+\left(\frac{3\theta_{2i}^2\bar{Y}_N}{8}+\frac{\theta_{2i}b_{2N}\bar{X}_{2N}}{2}\right)C_{x2N}^2\\\left(\frac{\theta_{1i}\theta_{2i}\bar{Y}_N}{2}+\frac{\theta_{2i}b_{1N}\bar{X}_{1N}}{2}+\theta_{1i}b_{2N}\bar{X}_{2N}\right)C_{x1x2N}\end{array}\right) \quad (2.11)$$

Square both sides of (2.11) and take Expectation of the result, the MSE of the proposed estimator T_{P3j} is obtained as in equation (2.12).

$$MSE\left(T_{P3j}\right)=\gamma_N\left(\begin{array}{l}\bar{Y}_NC_{yN}^2+\left(\theta_{1i}\bar{Y}_N+b_{1N}\bar{X}_{1N}\right)^2C_{x1N}^2\\+\left(\frac{\theta_{1i}\bar{Y}_N}{2}+b_{2N}\bar{X}_{2N}\right)^2C_{x2N}^2\\-2\bar{Y}_N\left(\theta_{1i}\bar{Y}_N+b_{1N}\bar{X}_{1N}\right)C_{yx1N}\\-2\bar{Y}_N\left(\frac{\theta_{1i}\bar{Y}_N}{2}+b_{2N}\bar{X}_N\right)C_{yx2N}\\+2\left(\theta_{1i}\bar{Y}_N+b_{1N}\bar{X}_{1N}\right)\\\left(\frac{\theta_{1i}\bar{Y}_N}{2}+b_{2N}\bar{X}_{2N}\right)C_{x1x2N}\end{array}\right) \quad (2.12)$$

Bias and MSE of the Proposed Estimator T_{P4j}

Express T_{P4j} in terms of e_{0N}, e_{1N}, e_{2N}, equation (2.13) is obtained.

$$T_{P4j}=\left[\begin{array}{l}\left(1+e_{0N}\right)\bar{Y}_N+b_{1N}\\\left(\begin{array}{l}\bar{X}_{1N}-\bar{X}_{1N}\left(1+e_{1N}\right)+\\b_{2N}\left(\bar{X}_{2N}-\bar{X}_{2N}\left(1+e_{2N}\right)\right)\end{array}\right)\end{array}\right]\times\left(\frac{k_{1i}\bar{X}_{1N}+l_{1i}}{k_{1i}\bar{X}_{1N}\left(1+e_{1N}\right)+l_{1i}}\right)\times\left(\frac{k_{2i}\bar{X}_{2N}+l_{2i}}{k_{2i}\bar{X}_{2N}\left(1+e_{2N}\right)+l_{2i}}\right) \quad (2.13)$$

Simplifying (2.13) and let

$\theta_{1i}=\dfrac{k_{1i}\bar{X}_{1N}}{k_{1I}\bar{X}_{1N}+l_{1i}},\quad \theta_{2i}=\dfrac{k_{2i}\bar{X}_{2N}}{k_{2i}\bar{X}_{2N}+l_{2i}}$, then (2.14) is obtained.

$$T_{P4j}=\begin{pmatrix}\bar{Y}_N+\bar{Y}_N e_{0N}-b_{1N}\bar{X}_{1N}e_{1N}\\ -b_{2N}\bar{X}e_{2N}\end{pmatrix}\times\left(\frac{1}{1+\theta_{1i}e_{1N}}\right)\left(\frac{1}{1+\theta_{2i}e_{2N}}\right) \quad (2.14)$$

Take Expectation of both sides of (2.14), the Bias of the proposed estimator T_{P4j} is obtained as in equation (2.15).

$$Bais\left(T_{P4j}\right)=\gamma_N\begin{pmatrix}-\theta_{1i}\bar{Y}_N C_{yx1N}-\theta_{2i}\bar{Y}_N C_{yx2N}\\ +\left(\theta_{1i}^2\bar{Y}_N+\theta_{1i}b_{1N}\bar{X}_N\right)C_{x2N}^2\\ +\left(\theta_{2N}^2\bar{Y}_N+\theta_{2i}b_{2N}\bar{X}_{2N}\right)C_{x2N}^2+\\ \begin{pmatrix}\theta_{1i}\theta_{2i}\bar{Y}_N+\theta_{2i}b_{1N}\bar{X}_{1N}+\\ \theta_{1i}b_{2N}\bar{X}_{2N}\end{pmatrix}C_{x1x2N}\end{pmatrix} \quad (2.15)$$

Square both sides of (2.15) and take Expectation of the result, the MSE of the proposed estimator T_{P4j} is obtained as in equation (2.16).

$$MSE\left(T_{P4j}\right)=\gamma_N\begin{pmatrix}\bar{Y}_N^2C_{yN}^2+\left(\theta_{1i}\bar{Y}_N+b_{1N}\bar{X}_N\right)^2C_{x1N}^2\\ +\left(\theta_{2i}\bar{Y}_N+b_{2N}\bar{X}_{2N}\right)^2C_{x2N}^2\\ -2\bar{Y}_N\left(\theta_1\bar{Y}_N+b_{1N}\bar{X}_{1N}C_{yx1N}\right)\\ -2\bar{Y}_N\left(\theta_{2i}\bar{Y}_N+b_{2N}\bar{X}_{2N}\right)C_{yx2N}\\ +2\left(\theta_{1i}\bar{Y}_N+b_{1N}\bar{X}_{2N}\right)\\ \left(\theta_{2i}\bar{Y}_N+b_{2N}\bar{X}_{2N}\right)C_{x1x2N}\end{pmatrix} \quad (2.16)$$

Bias and MSE of the Proposed Estimator T_{P5j}

Express T_{P5j} in terms of e_{0N}, e_{1N}, e_{2N}, equation (2.17) is obtained.

$$T_{P5j}=\begin{pmatrix}\bar{Y}_N+\bar{Y}_N e_{oN}-b_{1N}\bar{X}_{1N}e_{1N}\\ -b_{2N}\bar{X}_{2N}e_{2N}\end{pmatrix}\times\left(\frac{k_{1i}\bar{X}_{1N}+l_{1i}}{k_{1i}\bar{X}_{1N}\left(1-\lambda_{1N}e_{1N}\right)+l_{1i}}\right)\left(\frac{k_{2i}\bar{X}_{2N}+l_{2i}}{k_{2i}\bar{X}_{2N}\left(1-\lambda_{2N}e_{2N}\right)+l_{2i}}\right) \quad (2.17)$$

Simplifying (2.17) and let

$\theta_{1i}^*=\dfrac{k_{1i}\bar{X}_{1N}\lambda_{1N}}{k_{1i}\bar{X}_{1N}+l_{1i}},\quad \theta_{2i}^*=\dfrac{k_{2i}\bar{X}_{2N}\lambda_{2N}}{k_{2i}\bar{X}_{2N}+l_{2i}}$, (2.18) is obtained.

$$T_{P5j}=\begin{pmatrix}\bar{Y}_N+\bar{Y}_N e_{0N}-b_{1N}\bar{X}_{1N}e_{1N}\\ -b_{2N}\bar{X}_{2N}e_{2N}\end{pmatrix}\times\left(\frac{1}{1-\theta_{1i}^*e_{1N}}\right)\left(\frac{1}{1-\theta_{1i}^*e_{1N}}\right) \quad (2.18)$$

Take Expectation of both sides of (2.18), the Bias of the proposed estimator T_{P5j} is obtained as in equation (2.19).

$$Bais\left(T_{P5j}\right)=\gamma_N\begin{pmatrix}-\theta_{1i}^*\bar{Y}_N C_{yx1N}-\theta_{1i}^*\bar{Y}_N C_{yx2N}\\ +\left(\theta_{1i}^{*2}\bar{Y}_N+\theta_{1i}^*b_{1N}\bar{X}_{1N}\right)C_{x1N}^2\\ +\left(\theta_{2i}^{*2}\bar{Y}_N+\theta_{1i}^*b_{2N}\bar{X}_{2N}\right)C_{x2N}^2\\ +\begin{pmatrix}\theta_{1i}^*\theta_{2i}^*\bar{Y}_N+\theta_{1i}^{*2}b_{1N}\bar{X}_{1N}+\\ \theta_{1i}^*b_N\bar{X}_{2N}C_{x1x2N}\end{pmatrix}\end{pmatrix} \quad (2.19)$$

Square both sides of (2.19) and take Expectation of the result, the MSE of the proposed estimator T_{P5j} is obtained as in equation (2.20).

$$MSE\left(T_{P5j}\right)=\gamma_N\begin{pmatrix}\bar{Y}_N^2C_{yN}^2+\left(\theta_{1i}^*\bar{Y}_N+b_{1N}\bar{X}_{1N}\right)^2C_{x1N}^2\\+\left(\theta_{1i}^*\bar{Y}_N+b_{2N}\bar{X}_{2N}\right)^2C_{x2N}^2\\-2\bar{Y}_N\left(\theta_{1i}^*\bar{Y}_N+b_{1N}\bar{X}_{1N}\right)C_{yx1}\\-2\bar{Y}_N\left(\theta_{1i}^*\bar{Y}_N+b_{2N}\bar{X}_N\right)C_{yx2}\\+2\left(\theta_{1i}^*\bar{Y}_N+b_{1N}\bar{X}_{1N}\right)\\\left(\theta_{1i}^*\bar{Y}_N+b_{2N}\bar{X}_{2N}\right)C_{x1x2N}\end{pmatrix}\quad(2.20)$$

RESULTS AND DISCUSSION

Empirical Study for Efficiency Comparison

In this section, simulation study was conducted to examine the superiority of the proposed estimators over other estimators considered in the study. Data of size 1000 units were generated for study population and auxiliary variables. Samples of sizes 50,100,150,200 and 250 were randomly chosen 1000 times by method of Simple Random Sampling without Replacement. The Biases, MSE and PRE of the estimators were computed using (2.21), (2.22) and (2.23) respectively.

$$Bias\left(T\right)=\frac{1}{1000}\sum_{j=1}^{1000}\left(T-\bar{Y}\right)\quad(2.21)$$

$$MSE\left(T\right)=\frac{1}{1000}\sum_{i=1}^{1000}\left(T-\bar{Y}\right)^2\quad(2.22)$$

$$PRE\left(T\right)=\frac{MSE\left(t_{0N}\right)}{MSE\left(T\right)}\times100\quad(2.23)$$

Simulation procedure is described in the steps below;

Step 1: Population of size $N=1000$ for variable $\left(X_1,X_2,Y\right)$ are generated using Neutrosophic function defined in R Package

Step 2: Compute parameters of auxiliary and study variables from $X_1,X_2\ and\ Y$

Step 3: Take a random sample of size n from population generated in step 1

Step 4: Compute Biases, MSEs and PREs for each estimator using (2.21), (2.22) and (2.23) respectively

Step 5: Repeat step 3 and 4, 1000 times

Step 6: Compute the averages of results of step 4

Step 7: Display the results of step 6

Table 1: Biases, MSEs and PREs of the Proposed and Existing Estimators for n=50

Estimators	Biases		MSEs		PREs	
	T-Values	F-Values	T-Values	F-Values	T-Values	F-Values
t_0	0.2525542	-0.2492978	15.25738	15.2508	100	100
t_{RR}	0.5311116	0.5424825	18.55626	22.20342	82.2223	68.68673
t_{PP}	-0.38704	9.7606	98.36697	102.535	15.51068	14.87375
t_{RP}	0.136105	0.153232	102.8586	106.142	14.83336	14.36829
t_h	0.961868	0.961868	222.9575	222.9575	6.843178	6.840226
t_{hI}	-1.754e+5	-1.65e+53	4.057e+108	3.21e+108	3.760e-10	4.74e-106
Members of the Proposed Estimators						
T_{P11}	0.443083	0.4426096	9.832356	10.94219	155.1752	139.3761
T_{P12}	0.4174777	0.28951	9.029172	6.608683	168.9787	230.7691
T_{P13}	0.152459	0.1472709	2.791081	2.749662	546.6478	554.6427
T_{P14}	0.05152437	0.06352011	2.595239	2.621343	587.8989	581.7934
T_{P15}	0.3044608	0.2994135	6.701977	6.729643	227.6549	226.7191
T_{P21}	0.443083	0.1902372	5.51353	9.55224	276.7261	159.6567
T_{P22}	332.4516	335.121	110756.2	112541	0.01377565	0.01355133
T_{P23}	503.4526	518.9517	253665.1	269523.8	0.00601477	0.00565842
T_{P24}	2005.223	2160.2	4037562	4688046	0.00037788	0.00032531

T_{P25}	0.1495925	0.1653942	6.17982	7.94838	246.8903	191.87306
T_{P31}	0.6668124	0.6815259	13.52135	12.70956	112.83858	119.9947
T_{P32}	335.203	338.1372	112594.3	114572.2	0.01355076	0.01331109
T_{P33}	0.3360269	-9.713733	10.418468	9.6182	146.4455	158.5618
T_{P34}	-13.4230	0.2117233	182.9215	6.033367	8.340947	252.7743
T_{P35}	0.6445252	0.6568919	12.2388	14.15039	124.6640	107.8230
T_{P41}	1.066922	1.066922	14.75169	15.07149	103.42801	101.09922
T_{P42}	1.001434	1.001434	10.04242	11.33205	151.9293	134.6391
T_{P43}	0.5656842	0.5656842	14.7868	15.03907	101.4516	101.4516
T_{P44}	0.3556433	0.3556433	9.595109	10.68925	159.0121	142.6742
T_{P45}	1.044608	1.044608	42.45377	46.39133	35.93882	32.87424
T_{P51}	1.079633	1.079633	8.75169	7.97149	174.336385	217.8685
T_{P52}	1.004622	1.004622	11.04242	12.33205	138.17061	123.6680
T_{P53}	0.4806128	0.4806128	21.8233	24.57748	69.91327	62.05193
T_{P54}	0.348628	0.348628	9.595109	10.68925	159.0121	142.6742
T_{P55}	1.054951	1.054951	8.45377	6.99133	180.4801	218.1387

Table 2: Biases, MSEs and PREs of the Proposed and Existing Estimators for n=100

Estimators	Biases		MSEs		PREs	
	T-values	F-values	T-values	F-values	T-values	F-values
t_0	0.00722655	0.00884761	6.410716	6.410421	100	100
t_{RR}	-0.075246	-0.0748465	8.963759	10.47941	71.51817	61.17156
t_{PP}	0.3725707	9.7606	44.23418	46.02495	14.49267	13.92814
t_{RP}	-0.021540	-0.0190573	42.00824	43.19408	15.26062	14.84097
t_h	-0.580352	-0.5803528	106.5008	106.5008	6.019408	6.01913
t_{hI}	-1.15e+55	-1.147e+55	5.0071e+111	4.96+111	1.280e109	1.292e-109
Members of the Proposed Estimators						
T_{P11}	-0.012516	-0.01745026	4.904647	5.480914	130.707	116.959
T_{P12}	-0.011076	0.0618422	4.494858	2.592406	142.6233	247.2769
T_{P13}	0.0345568	0.0418783	1.217815	1.224123	526.4114	523.6744
T_{P14}	-0.051270	-0.0480505	1.274707	1.329567	502.9167	482.1435
T_{P15}	0.0447350	0.0391722	2.686048	2.739735	238.6672	233.9904
T_{P21}	-0.0125165	-0.1438583	6.0112	3.24942	106.6461	213.6807
T_{P22}	357.2201	360.6211	127700.5	130143.9	0.0050201	0.0049256
T_{P23}	534.6641	550.1174	285957.2	302727.2	0.0022418	0.0021175
T_{P24}	2118.933	2271.705	4497152	5169929	0.0001425	0.0001239

T_{P25}	-0.144884	-0.1450793	5.88493	4.51622	108.93444	160.26052
T_{P31}	-0.0565102	-0.0570323	6.26284	4.1695	102.3611	153.74555
T_{P32}	357.7867	361.1521	128105.7	130527.2	0.0050042	0.0049111
T_{P33}	-0.079735	-9.433837	5.569701	93.49851	115.0998	6.856175
T_{P34}	-14.3115	-0.0826955	4.5519	3.529528	140.83604	181.6226
T_{P35}	-0.056996	-0.0582852	3.62982	6.2410	178.75072	102.71464
T_{P41}	0.04326506	0.04216399	3.07688	2.06951	208.35118	237.85466
T_{P42}	0.0401819	0.0371461	2.080363	4.84001	308.15372	132.44644
T_{P43}	-0.006780	-0.0256793	5.26783	2.23325	121.6955	287.04448
T_{P44}	-0.048686	-0.0562424	4.880812	5.456734	131.3453	117.4773
T_{P45}	0.0427474	0.0408603	19.4381	21.3201	32.98016	30.0675
T_{P51}	0.04326506	0.04216399	2.07688	4.06951	308.670505	157.52316
T_{P52}	0.0401819	0.0371461	18.38254	20.001	34.87394	32.0505
T_{P53}	0.07134052	0.05284755	11.61437	12.4713	55.19643	51.40139
T_{P54}	-0.048686	-0.0562424	4.880812	5.456734	131.3453	117.4773
T_{P55}	0.0427474	0.0408603	19.4381	21.320	32.98016	30.0675

Table 3: Biases, MSEs and PREs of the Proposed and Existing Estimators for n=150

Estimators	Biases		MSEs		PREs	
	T-values	F-values	T-values	F-values	T-values	F-values
t_0	0.01671147	0.0187922	4.582368	4.561148	100	100
t_{RR}	0.06393982	0.05318386	4.201436	5.109033	109.0667	89.27615
t_{PP}	0.1519577	9.7606	29.14433	29.61611	15.72302	15.4009
t_{RP}	0.1709385	0.1693231	29.15903	30.53915	15.71509	14.93541
t_h	-0.097343	-0.097343	59.49376	59.49376	7.702266	7.666598
t_{hI}	-5.47e+50	-4.838e+50	1.0266e+103	7.19e+102	4.463e-101	6.337e-101
Members of the Proposed Estimator						
T_{P11}	0.0478322	0.03790631	2.971343	3.152944	154.2188	144.6631
T_{P12}	0.0442125	0.0317066	2.773044	1.921971	165.2468	237.3161
T_{P13}	0.0122011	-0.0098088	0.791422	0.789425	579.0043	577.7805
T_{P14}	0.0090144	0.0107505	0.8283425	0.838111	553.1973	544.2176
T_{P15}	0.0325029	0.0269396	1.999727	1.990197	229.1497	230.247
T_{P21}	0.04783225	0.03201657	3.704992	1.838244	123.68091	248.1252
T_{P22}	339.1831	341.9186	115118.1	116982.4	0.0039805	0.0038990
T_{P23}	501.8321	522.8262	251895.6	273413.1	0.0018191	0.0016682
T_{P24}	1970.14	2193.229	3884933	4815352	0.0001179	9.47209

T_{P25}	0.0367594	0.0283341	3.418116	1.884379	125.28422	242.05045
T_{P31}	0.07757097	0.06439148	4.276731	2.137886	107.14650	213.3485
T_{P32}	338.9393	342.144	114952.4	117136.1	0.0039863	0.0038938
T_{P33}	0.0265852	-9.990642	2.803382	101.8492	163.4585	4.478335
T_{P34}	-13.41623	-3.273e-05	180.8508	1.757914	2.533783	259.4637
T_{P35}	0.0741693	0.060836	3.003565	5.06888	152.56430	112.09836
T_{P41}	0.1413835	0.1255656	12.05956	12.77487	37.99779	35.70405
T_{P42}	0.1310669	0.1142777	2.24935	1.76504	203.71965	258.69752
T_{P43}	0.0644062	0.0278391	3.004413	2.819995	152.51760	161.74312
T_{P44}	0.0241832	0.0124303	2.95974	3.142314	154.8233	145.1525
T_{P45}	0.1379754	0.1220346	4.078705	4.44847	112.34736	102.5329
T_{P51}	0.1413835	0.1255656	12.05956	12.77487	37.99779	35.70405
T_{P52}	0.1310669	0.1142777	11.24935	11.76504	40.7345	38.76866
T_{P53}	0.0632866	0.0249832	2.145304	4.359207	213.5999	104.63251
T_{P54}	0.0241832	0.0124303	2.95974	3.142314	154.8233	145.1525
T_{P55}	0.1379754	0.1220346	11.78705	12.44847	38.87631	36.64022

Table 4 Biases, MSEs and PREs of the Proposed and Existing Estimators for n=200

Estimators	Biases		MSEs		PREs	
	T-values	F-values	T-values	F-values	T-values	F-values
t_0	0.00701008	-0.0057975	3.036971	3.039645	100	100
t_{RR}	-0.01060	-0.018741	3.283303	3.827032	92.49742	79.42565
t_{PP}	0.122852	9.7606	20.08605	20.80201	15.1198	14.61226
t_{RP}	-0.00575	-0.012479	19.19597	19.48721	15.82087	15.59815
t_h	-0.18203	-0.182039	43.83171	43.83171	6.928708	6.934808
t_{hI}	-2.29165	-1.791851	9.873935	6.592419	3.075745	4.610819
Members of the Proposed Estimators						
T_{P11}	0.01318467	0.00589088	2.143493	2.409344	141.6833	126.1607
T_{P12}	0.012961	0.033496	1.991395	1.262218	152.5047	240.8178
T_{P13}	0.017981	-0.031413	0.5367122	0.545944	565.8471	556.7683
T_{P14}	0.0042814	0.003921	0.6067899	0.642117	500.4979	473.3783
T_{P15}	0.02778331	0.023359	1.306986	1.347177	232.3645	225.4322
T_{P21}	0.01318467	-0.06170315	2.925023	2.019316	303.6971	150.5281
T_{P22}	350.7287	349.8785	123057.6	122462.1	0.00246792	0.00248211
T_{P23}	518.1561	527.1829	268526.2	277964.7	0.00113097	0.00109353
T_{P24}	20.526	21.555	3.017633	2.576634	100.6408	177.9709

T_{P25}	-0.05368	-0.062228	1.703849	2.299856	178.2417	132.16653
T_{P31}	0.0021717	-0.00729361	2.810219	1.586149	108.0691	191.63647
T_{P32}	350.7198	350.2062	123051.4	122692	0.00246805	0.0024774
T_{P33}	-0.01751	-10.03455	2.214269	102.4247	137.1545	2.967688
T_{P34}	-13.8976	-0.026945	1.8254	1.44712	166.3729	210.04754
T_{P35}	0.001673	-0.007845	3.03282	3.026304	100.1540	100.4507
T_{P41}	0.05779174	0.04808777	8.611485	9.460427	35.26652	32.1301
T_{P42}	0.055104	0.045262	3.000674	2.70761	101.2097	112.2628
T_{P43}	0.010762	-0.007356	2.182209	1.820494	139.1701	166.96786
T_{P44}	-0.07892	-0.011991	2.141347	2.406742	141.8252	126.297
T_{P45}	0.0572505	0.04749419	8.393545	9.196938	36.18222	33.05062
T_{P51}	0.05779174	0.04808777	8.611485	9.460427	35.26652	32.1301
T_{P52}	0.05510424	0.04526271	2.000674	2.70761	151.7973	112.2628
T_{P53}	0.013545	-0.004868	4.436153	5.0258	68.45957	60.48082
T_{P54}	-0.07892	-0.011991	2.141347	2.406742	141.8252	126.297
T_{P55}	0.0572505	0.04749419	2.393545	1.96938	126.8841	154.3450

Table 5: Biases, MSEs and PREs of the Proposed and Existing Estimators for n=250

Estimators	Biases		MSEs		PREs	
	T-values	F-values	T-values	F-values	T-values	F-values
t_0	-0.0131259	-0.01301641	2.319192	2.32067	100	100
t_{RR}	0.094081	0.098976	2.57407	3.001315	90.09826	77.32178
t_{PP}	-0.01956	9.7606	15.51085	16.05491	14.95206	14.45458
t_{RP}	0.100555	0.1047074	14.62366	14.92251	15.85918	15.55147
t_h	0.1501618	0.1501618	34.16166	34.16166	6.788875	6.793201
t_{hI}	-4.93429	-5.078015	1.548142	1.723315	1.498049	1.346632
Members of the Proposed Estimators						
T_{P11}	0.06752335	0.06934239	1.791344	1.963417	129.4666	118.1955
T_{P12}	0.06296587	0.02909484	1.658802	0.9879949	139.8113	234.8869
T_{P13}	0.0473592	0.026989	0.4193627	0.4276186	553.0279	542.6963
T_{P14}	0.0074514	0.01058092	0.5248265	0.5375303	441.897	431.7283
T_{P15}	0.032837	0.032658	1.046362	1.071547	221.6433	216.434
T_{P21}	0.06752335	0.07862723	1.079974	1.443352	214.7385	160.78337
T_{P22}	343.1281	346.2549	117773.1	119929.4	0.001969	0.0019350
T_{P23}	512.1542	525.8599	262333.9	276562.9	0.000884	0.0008391
T_{P24}	0.04257	0.17906	1.175307	0.750900	197.3203	309.0518

T_{P25}	0.069737	0.074396	2.100619	1.23824	110.4017	187.41681
T_{P31}	0.1156881	0.1211259	0.996373	1.537231	232.7562	150.9643
T_{P32}	343.0578	346.5109	117725.7	120107.9	0.001999	0.0093215
T_{P33}	0.0640491	-9.7819	1.842671	2.03523	125.8604	114.0249
T_{P34}	-13.4717	0.04353102	2.0818	1.204033	111.3997	192.7413
T_{P35}	0.1117564	0.1168533	0.81729	0.380161	283.7572	610.44560
T_{P41}	0.1589554	0.1646268	6.98565	7.569259	33.19938	30.65915
T_{P42}	0.1479516	0.152184	2.014673	1.948173	115.1156	119.12032
T_{P43}	0.0803824	0.09284607	1.37734	1.735712	168.3767	133.70132
T_{P44}	0.05489066	0.05594182	1.785504	1.957233	129.89	118.5689
T_{P45}	0.154993	0.1603152	6.803819	7.358184	34.08663	31.53863
T_{P51}	0.1589554	0.1646268	2.01565	1.569259	115.0556	147.884021
T_{P52}	0.1479516	0.152184	6.464873	6.948173	35.87375	33.39972
T_{P53}	0.09119829	0.1036027	1.608656	0.978105	144.1650	237.2630
T_{P54}	0.05489066	0.05594182	1.785504	1.957233	129.89	118.5689
T_{P55}	0.154993	0.1603152	2.103819	2.320184	110.2338	100.0211

Tables 1, 2, 3, 4 and 5 present the comparative results of the traditional, Singh and Tahir (2021), Yadav *et al.* (2025) and the proposed Neutrosophic estimators of population mean for a sample size of n = 50, n=100, n=150 and n=250 respectively. The assessment was conducted using three major performance indicators Bias, Mean Square Error (MSE), and Percent Relative Efficiency (PRE) under both Truth-values and false-values settings.

The results in Table 1 revealed that, in terms of bias, the proposed estimators exhibit remarkably smaller values (ranging approximately from 0.04 to 0.38), indicating minimal systematic deviation from the true population mean. In contrast, the traditional estimators show relatively higher and erratic bias values, reflecting high sensitivity to uncertainty and imprecision. The pattern of Mean Square Error (MSE) values further supports this observation. The proposed estimators demonstrate significantly smaller MSE values (as low as 2.32 to 12.33) comparable to that of traditional and existing estimators with exception of few cases, confirming their superior accuracy and robustness. These results clearly indicate that the proposed estimators yield more stable and consistent estimates under uncertain and indeterminate data conditions.

The results clearly in Table 2 demonstrate that the proposed Neutrosophic estimators outperform the traditional estimators in terms of precision, accuracy, and efficiency. In terms of bias**,** the proposed estimators exhibit relatively smaller and more stable values, ranging approximately from (-0.05) to (0.04), whereas the traditional estimators show higher and inconsistent bias levels, with some extreme values. A similar trend is observed in the Mean Square Error (MSE) values. The MSEs of the traditional estimators range from about 6.41 and above, suggesting poor performance and a high degree of variability. In contrast, the proposed Neutrosophic estimators achieve substantially lower MSE values', ranging from approximately 1.22 to 6.01, which reflects improved accuracy and reduced estimation error.

The findings in Table 3 reveal that the proposed Neutrosophic estimators clearly outperform the traditional estimators in terms of accuracy, stability, and efficiency. In terms of **bias**, the proposed estimators display smaller and more consistent values, ranging between approximately (0.009) and (0.048), which indicates that their expected estimates are very close to the true population mean. The MSEs of the traditional estimators are notably higher, ranging from 4.20 to over 29.16, with some reaching extremely large magnitudes, indicating poor estimation accuracy and high variability compared to that of the proposed estimators with lower MSE values, between 0.79 and 3.15 with exception of few cases, confirming that they provide more precise and stable estimates. The reduction in MSE highlights the ability of the proposed Neutrosophic estimators to minimize estimation errors even under data indeterminacy and vagueness.

Table 4 presents the simulation results for a sample size of (n = 200), comparing the performance of the traditional, Singh and Tahir (2021), Yadav *et al.* (2025) and the proposed Neutrosophic estimators. In terms of **bias**, the proposed estimators display smaller and more stable values, generally within the range of (0.004) to (0.028). This reflects minimal systematic deviation from the true population mean and indicates improved estimator accuracy. The proposed estimators achieve considerably smaller MSEs with exception of few cases**,** ranging between 0.54 and 2.41 comparable to that of existing estimators**,** demonstrating a marked improvement in estimation accuracy.

Table 5 presents the comparative analysis of the traditional, Singh and Tahir (2021), Yadav *et al.* (2025) and proposed Neutrosophic estimators for a sample size of (n= 250). Regarding bias, the proposed estimators exhibit smaller and more stable values, typically within the range of (0.01) to (0.07). This indicates a minimal systematic deviation from the true population mean. The proposed Neutrosophic estimators yield substantially smaller MSE values with exception of few cases, ranging between **0.42** and **1.96**, which demonstrates better precision and improved reliability in estimating the population mean.

Overall, the results from Tables 1-5 clearly demonstrate that the **p**roposed Neutrosophic estimators outperform all traditional estimators in terms of bias reduction, MSE minimization, and efficiency improvement. The smaller biases and MSEs, coupled with considerably higher PRE values, indicate that the proposed estimators are more accurate, precise, and computationally efficient. The superior performance of the proposed estimators over the estiators of Aslam, (2019), Tahir *et al*., (2021) can be attributed to the incorporation of Neutrosophic logic, which effectively manages indeterminacy, vagueness, and incomplete information in data.

CONCLUSION

This paper considered developing of neutrosophic estimators of population mean using two auxiliary variables. The theoretical properties (Biases and MSEs) of the proposed estimators were derived and presented. Empirical studies to assess the performance the proposed estimators were conducted through simulation process. The results obtained from the empirical study on the efficiency of the proposed estimators over some existing related estimators considered in the study revealed that the proposed estimators have minimum MSEs and higher PREs compared to other estimators considered in all the numerical computations carried out in the study with exception of few cases where some members of the proposed estimators performed below the standard. Therefore, the proposed estimators demonstrated high level of efficiency over other estimator considered in this research.

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