Journal of Basics and Applied Sciences Research (JOBASR)
ISSN (print): 3026-9091, ISSN (online): 1597-9962
Volume 3(6) November 2025
DOI: https://dx.doi.org/10.4314/jobasr.v3i6.18

Non-autonomous Equations of Restricted Three-Body Problem with Variable Masses and Zonal Harmonics up to J4

Taura, Joel John[1*], Leke, Oni.[2] & Singh, Jagadish[3]

[1]Department of Mathematics and Statistics, Federal University of Kashere, Gombe-State, Nigeria
[2]Department of Mathematics, College of Physical Science, Joseph Sarwuan Tarka, University, P.M.B. 2373, Makurdi, Benue-State Nigeria
[3]Department of Mathematics, Faculty of Science, Ahmadu Bello University, Zaria, Kaduna-State, Nigeria
[*]Corresponding Author Email: taurajj@yahoo.com

Keywords:
Gylden-Mestschersky problem;
RTBP;
Variable Masses;
Oblateness;
Zonal Harmonics; J_4

ABSTRACT
This paper investigates the derivations of the time-dependent equations of motion of a test particle in the frame of the R3BP with variable masses and zonal harmonics. The motion and mass variations of the primaries are described by the Gylden-Mestschersky problem (GMP) and the unified Mestschersky law (UML), respectively, with further assumptions that the oblateness of the bigger primary varies with zonal harmonics coefficients up to J_4 terms. The non-autonomous equations of the test mass in a reference frame rotating are derived using the Hamiltonian method. These equations are DE with variable coefficients and are defined by the oblateness of the bigger body with zonal harmonics coefficients up to J_4, the angular velocity of revolution and the masses of the primaries. This study will in no doubt expand the knowledge base of celestial mechanics and will allow for more extensions with applications to space missions.

INTRODUCTION

The restricted three-body problem (RTBP) is a model description which studies motion of a third body of infinitesimal mass in the gravitational environment of two main bodies called primaries. The more massive body is called the bigger or first primary while the less massive one is called the smaller or secondary body. The motion and gravitational force of the infinitesimal body do not affect the primaries (Szebehely 1967). The RTBP has had major applications in various scientific fields, such as in celestial mechanics, chaos theory, molecular physics, astrodynamics, astrophysics and galactic dynamics (Singh and Leke 2014).

The RTBP are governed by non-integrable differential equations, hence equilibrium solutions are needed to get insights into the dynamical predictions of the infinitesimal mass. These solutions are obtained when the velocity and acceleration components are zero, and are referred to as the equilibrium or libration or Lagrangian points. For the classical RTBP there exists three collinear and two triangular points. The collinear points are located on the line joining the primaries while the triangular points form two equilateral triangles with the primaries (Szebehely 1967). The collinear points are unstable while the triangular points are conditionally stable (Szebehely 1967).

In recent years focus has been drawn to the study of the RTBP under different characterizations of the main bodies and the infinitesimal mass. Some of such characterizations include radiation pressure of one or both primaries, oblateness or triaxiality of one or both primaries (see Singh and Ishwar 1999, AbdulRaheem and Singh 2006, Singh and Leke 2014), inclusion of a disk in the configuration of the RTBP (Singh and Taura 2015, Leke and Singh 2023). Others have considered effects of zonal harmonics (Bury and McMahon 2020, Gyegwe et al 2025), while some have discussed mass variation effects (see Singh and Leke 2010, 2012, 2013; Leke and Mmaju 2023, Leke and Orum 2024, Leke et al 2024& 2025).

The classical RTBP traditionally presumes that the masses of the primary bodies remain constant. However, observations of stellar absorption phenomena prompted researchers to extend the RTBP to include variable mass systems. The formulation of the RTBP with variable masses is significant in both astronomical and engineering applications. For instance, in analyzing the motion of spacecraft near comets or asteroids undergoing mass loss due to surface outgassing, studying binary star systems experiencing mass transfer, and examining the Earth–Moon system during episodes of lunar mass discharge.

How to cite this article: Taura, J.J., Leke, O. & Singh, J. (2025). Non-autonomous Equations of Restricted Three-Body Problem with Variable Masses and Zonal Harmonics up to J4. *Journal of Basics and Applied Sciences Research*, *3*(6), 170-176. https://dx.doi.org/10.4314/jobasr.v3i6.18

Incorporating mass variation into the RTBP has led to new insights: Bekov (1988) identified additional equilibrium points, referred to as coplanar equilibrium points, while Singh and Leke (2010) investigated the stability of the photogravitational RTBP with variable masses. Other scientists that have carried out researches on the R3BP with variable masses under different classifications includes Leke and Singh (2023), Ibraimova et al (2023), Leke and Orum (2024), Gao et al (2024), and Leke et al. (2024, 2025).

Motivated by the extensive applications of the RTBP with variable masses, this study aims to formulate and analyze the equations of motion of the RTBP incorporating both variable masses and the oblateness of the smaller primary, expressed through zonal harmonic terms up to J_4. The variations in the masses of the primaries and the oblateness of the bigger primary are assumed to evolve with time according to the unified Mestschersky law (UML). The inclusion of zonal harmonic effects in the variable-mass RTBP framework enhances the model's realism by providing a more precise representation of the actual dynamical behavior of celestial bodies, thereby improving the accuracy of predictive analyses. For instance, the model proposed by Bekov (1988) may not adequately capture the motion of a test particle within the gravitational field of irregularly shaped bodies such as asteroids. Similarly, the treatment of oblateness up to the J_4 term by Abouelmagd (2012) may not have fully revealed the complete set of equilibrium points or their stability characteristics. Hence, it is both timely and appropriate to examine the dynamical behavior of the RTBP with variable masses and time-dependent oblateness of the larger primary, incorporating zonal harmonic coefficients up to the J_4 term. The natural starting point for such an investigation is the derivation of the governing dynamical equations. Hence, our main contribution in the present work is to derive the equations of motion of the RTBP under effect of varying masses and zonal harmonics in the oblateness of the bigger primary. Consequently, we shall consider the influence of even zonal harmonic parameters up to J_4 for the bigger primary.

This paper is structured as follows: the introduction is provided in current Section. The dynamical equation of two bodies with variable masses (Gylden-Mestschersky Problem (GMP), Gylden 1884, Mestschersky 1902) is presented in Section 2, while the potential energy of the infinitesimal body is presented in Section 3. The equations of motion of the time-dependent system are derived in Section 4. Section 5, gives the results, while the discussion and conclusion are given in Section 6. These derivations are extension of the dynamical equations of Abouelmagd (2012) with the consideration that the masses of the primaries and zonal harmonics in the oblateness of the bigger primary change with time. Also, it could also be viewed as an extension of the work of Singh and Leke (2012) by considering oblateness of the bigger primary with zonal harmonic coefficient to J_4 terms in the absence of perturbations in the Coriolis and centrifugal forces. Additionally, it could be seen as an extension of the work done by Bekov (1988) by introducing the oblateness of the bigger primary with zonal harmonic of $J_2$and J_4 terms.

MATERIALS AND METHODS

2.1 Equation of Motion of Two- Body with Variable Masses (GMP)

We start the process of our derivations of the dynamical equations with the two-body problem (2BP), which is the initial point for virtually all reference books in the expanse of astrodynamics. The fundamental set up illustrates the motion of two point-masses under mutual gravitational attraction, and is given by

$$\frac{d\vec{v}}{dt} = -\frac{\mu}{r^2}\frac{\vec{r}}{r} \tag{1}$$

where $\vec{v}$ is the velocity, μ is the gravitational constant G multiplied by the sum of the masses, and $\vec{r}$ is the distance between the bodies and connects the angular velocity θ and constant of the area integral C by the relation:

$$r^2\dot{\theta} = C \tag{2}$$

Now, the 2BP with variable masses, which is similar to the classical 2BP with constant masses, is given by the equation

$$\ddot{\vec{r}} = -\frac{G(m_1 + m_2)\vec{r}}{r^3} \tag{3}$$

Equation (1) is identical to equation (3) with the difference that in equation (3), the total of the masses is a certain function of time. This equation is called the Gylden-Mestschersky problem (GMP).

Mestschersky (1902), transformed the GMP (3) to (1) by introducing new variables and time using a conversion. These equations that converts to constant system of equations was later referred to as the Mestschersky (1902) transformation (MT) and is expressed as

$$x = \xi R(t), y = \eta R(t), z = \zeta R(t), dt/d\tau = R^2(t)$$
$$r_i = \rho_i R(t), r = \rho_{12} R(t) \ \ (i = 1,2) \tag{4}$$

where $R(t) = \sqrt{\alpha t^2 + 2\beta t + \gamma}; \xi, \eta, \zeta, \tau$ are the newly introduced variables and ρ_{12} is constant.

Further, Mestschersky (1952) devised a law which was later referred to as the unified Mestschersky law (UML). This law asserts that the masses and their sum vary in the same proportion in such a way that

$$\mu(t)=\frac{\mu_0}{R(t)}, \mu_1(t)=\frac{\mu_{10}}{R(t)}, \mu_2(t)=\frac{\mu_{20}}{R(t)} \quad (5)$$

where

$\mu_1(t)=Gm_1(t), \mu_2(t)=Gm_2(t), \mu(t)=\mu_1(t)+\mu_2(t), \mu_{10}$ and μ_{20} are constants.

$$r\mu=\kappa C^2 \quad (6)$$

is the particular integral of the GMP where κ is a constant.

3 The Potential Energy of the Infinitesimal Body

In free space the gravitational potential external to a body which is oblate with its mass apportioned symmetrically around its equator, can be expanded in terms of Legendre polynomials in the pattern (Abouelmagd 2012, Bury and McMahon 2020)

$$V(r)=-\frac{GM}{r}\left[1-\sum_{k=1}^{\infty}\left(\frac{R}{r}\right)^k J_k P_k \cos\theta\right] \quad (7)$$

where, G refers to the gravitational constant, R is the radius and r is the distance from the body to the test particle, while J_k is the kth zonal harmonic term, and θ is the angle between the body's symmetry axis and the vector to a test particle. The terms $P_k \cos\theta$ are the Legendre polynomials, given by

$$P_k(x)=\frac{1}{2^k(k)!}\frac{d^k}{dx^k}\left(x^2-1\right)^k \quad (8)$$

Since the present study is concerned with planar problem, assuming the equatorial plane of both primaries coincide with the plane of motion, then with $\theta=90^0$, Equation (7) becomes

$$V(r)=-\frac{GM}{r}\left[1+\frac{J_2R^2}{2r^2}-\frac{3J_4R^4}{8r^4}+\frac{5J_6R^6}{16r^6}\right] \quad (9)$$

the potential energy of the infinitesimal body, under the influence of varying oblateness of the bigger primary with zonal harmonics up to J_4 term has the form

$$U=-Gm_3\left[\frac{m_1}{r_1}+\frac{m_2}{r_2}+\frac{m_1A_1(t)}{2r_1^3}-\frac{5m_1A_2(t)}{8r_1^3}\right] \quad 10)$$

Where $r_i=\sqrt{(x-x_i)^2+y^2+z^2}\quad (i=1,2) \quad (11)$

$$A_1(t)=J_2R_1^2 \text{ and } A_2(t)=J_4R_1^4 \quad (12)$$

$A_i(i=1,2)$ are the time-dependent oblateness up to J_2 and J_4, respectively, of the bigger primary while R_1 is the time-dependent mean radii of the bigger primary. r_1 and r_2 are the distances of the infinitesimal mass from the bigger and smaller primaries, respectively.

4 Equations of Motion of the Infinitesimal Mass

To deduce the dynamical equations of the RTBP of our formulation, we consider a synodic frame of reference $0xyz$, with 0 as the origin. We let m_1 and m_2 be the masses of the massive bodies and m_3 be the mass of the dust grain. We have taken the line joining m_1 and m_2 as the $x-$axis. The coordinates of m_1 and m_2 are $(x_1,0,0)$ and $(x_2,0,0)$ respectively, and that of the dust grain is (x,y,z). Here we suppose that the masses of the bodies vary with respect to time with the consideration that the bigger body is an oblate spheroid with zonal harmonics terms up to J4. The radius vector from m_3 to m_1 is r_1 while that between m_3 and m_2 is r_2, and the distance between the bodies is r. Also, let ω be the angular velocity as illustrated in Figure 1

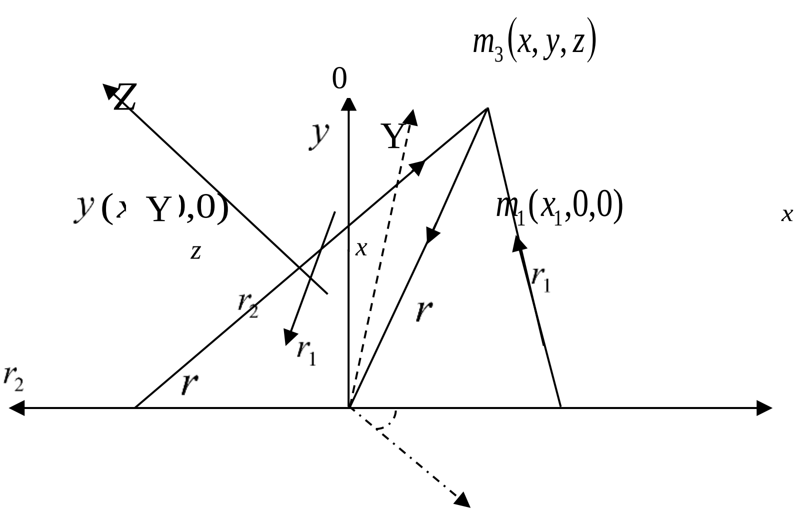

Fig. 1 Description of the RTBP

We consider same formulation by Abouelmagd (2012) with the assumptions that the masses and oblateness of the bigger primary change according to the unified Mestschersky law (UML)

Now, the kinetic energy of the infinitesimal mass in the rotating frame of reference $0xyz$ is given by

$$T=\frac{1}{2}m_3\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)+ m_3\omega\left(x\dot{y}-y\dot{x}\right)+\frac{1}{2}m_3\left(x^2+y^2\right)\omega^2 \quad (13)$$

Now let p_x, p_y and p_z be the generalized components of momentum then,

$$p_x=\frac{\partial T}{\partial \dot{x}}=m_3\left(\dot{x}-\omega y\right), p_y=\frac{\partial T}{\partial \dot{y}}= m_3\left(\dot{y}+\omega x\right), p_z=\frac{\partial T}{\partial \dot{z}}=m_3\dot{z} \quad (14)$$

This implies

$$\dot{x}=\frac{p_x}{m_3}+\omega y, \dot{y}=\frac{p_y}{m_3}-\omega x, \dot{z}=\frac{p_z}{m_3} \quad (15)$$

Substituting equations of system (14) in (13) and simplifying, we at once have

$$T=\frac{1}{2m_3}(p_x^2+p_y^2+p_z^2) \quad (16)$$

Now, the Hamiltonian H is given by

$$H=\frac{1}{2}m_3\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)- \frac{1}{2}m_3\omega^2\left(x^2+y^2\right)+U \quad (17)$$

Using system (15) in equation (17), we get

$$H=\frac{1}{2m_3}(p_x^2+p_y^2+p_z^2)+\omega\left(yp_x-xp_y\right)+U$$

(18)

Now, from the Hamiltonian canonical equations, we have

$$\dot{p}_x=-\left(-\omega p_y\right)-\frac{\partial U}{\partial x},$$

$$\dot{p}_y=-\omega p_y-\frac{\partial U}{\partial y},$$

$$\dot{p}_z=-\frac{\partial U}{\partial z} \quad (19)$$

Since the primaries move within the frameworks of the GMP with their masses changing with time as defined by the UML then p_x, p_y, p_z and the angular velocity ω will all depend on time. So, we differentiate system (14) w.r.t time and compare with equations (19), to get

$$m_3\left(\ddot{x}-\omega\dot{y}-\dot{\omega}y\right)=\omega p_y-\frac{\partial U}{\partial x},$$

$$m_3\left(\ddot{y}+\omega\dot{x}-\dot{\omega}x\right)=-\omega p_x-\frac{\partial U}{\partial y}, \quad (20)$$

$$m_3\ddot{z}=-\frac{\partial U}{\partial x}$$

Substituting equations of system (14) in (20), we get

$$m_3\left(\ddot{x}-\omega\dot{y}-\dot{\omega}y\right)=\omega m_3\left(\dot{y}+\omega x\right)-\frac{\partial U}{\partial x}$$

$$m_3\left(\ddot{y}-\omega\dot{x}-\dot{\omega}x\right)=-\omega m_3\left(\dot{x}-\omega y\right)-\frac{\partial U}{\partial y} \quad (21)$$

$$m_3\ddot{z}=-\frac{\partial U}{\partial z}$$

If we differentiate equation (10) w.r.t x, y and z, respectively, and substitute in (21) while multiplying the results throughout by $\frac{1}{m_3}$, we get

$$\ddot{x}=2\omega\dot{y}+\omega^2x+\dot{\omega}y-\frac{\mu_1\left(x-x_1\right)}{r_1^3} -\frac{\mu_2\left(x-x_2\right)}{r_2^3}-\frac{3}{2}\frac{\mu_1A_1\left(t\right)\left(x-x_1\right)}{r_1^5} +\frac{15}{8}\frac{\mu_1A_2\left(t\right)\left(x-x_1\right)}{r_1^7}$$

$$\ddot{y}=-2\omega\dot{x}+\omega^2y-\dot{\omega}x- \frac{\mu_1y}{r_1^3}-\frac{\mu_2y}{r_2^3}-\frac{3}{2}\frac{\mu_1A_1\left(t\right)y}{r_1^5} +\frac{15}{8}\frac{\mu_1A_2\left(t\right)y}{r_1^7}$$

(22)

$$\ddot{z}=-\frac{\mu_1z}{r_1^3}-\frac{\mu_2z}{r_2^3}-\frac{3}{2}\frac{\mu_1A_1\left(t\right)z}{r_1^5}+\frac{15}{8}\frac{\mu_1A_2\left(t\right)z}{r_1^7}$$

Where

$$r_1^2 = (x - x_1)^2 + y^2 + z^2, \quad r_2^2 = (x - x_2)^2 + y^2 + z^2$$

$$\omega^2(t) = \frac{\mu(t)}{\kappa}\left[\frac{1}{r^3} + \frac{3}{2r^5}A_1 - \frac{15}{8r^7}A_2\right] \qquad (23)$$

RESULTS AND DISCUSSION

Equations (22) model the motion of an infinitesimal body within the gravitational field of two larger primary bodies. The model is set in a rotating coordinate system centered on the system's center of mass and accounts for the changing mass and oblateness (up to the J4 zonal harmonic) of the larger primary body over time. These factors introduce additional terms not found in the earlier equations of Bekov (1988) and Singh and Leke (2010-2013).

The systems differ from those of Singh and Leke (2012) due to zonal harmonics terms up to J_4 in the oblateness of the bigger primary. If the zonal harmonics in the oblateness of the bigger primary up to J_4 are ignored, we have $A_2 = 0$, and the autonomized systems reduce to

$$\ddot{x} = 2\omega\dot{y} + \omega^2 x + \dot{\omega} y - \frac{\mu_1(x - x_1)}{r_1^3} - \frac{\mu_2(x - x_2)}{r_2^3} - \frac{3}{2}\frac{\mu_1 A_1(t)(x - x_1)}{r_1^5}$$

$$\ddot{y} = -2\omega\dot{x} + \omega^2 y - \dot{\omega} x - \frac{\mu_1 y}{r_1^3} - \frac{\mu_2 y}{r_2^3} - \frac{3}{2}\frac{\mu_1 A_1(t) y}{r_1^5} \qquad (24)$$

$$\ddot{z} = -\frac{\mu_1 z}{r_1^3} - \frac{\mu_2 z}{r_2^3} - \frac{3}{2}\frac{\mu_1 A_1(t) z}{r_1^5}$$

where

$$\omega^2(t) = \frac{\mu(t)}{\kappa r^3}\left[1 + \frac{3}{2r^2}A_1\right] \qquad (25)$$

which is the time-dependent dynamical equations of Singh and Leke (2012) in the absence of small perturbations in the Coriolis and centrifugal forces.

If further, we ignore the oblateness up to J_2, we have $A_1 = 0$ and equations reduce to

$$\ddot{x} = 2\omega\dot{y} + \omega^2 x + \dot{\omega} y - \frac{\mu_1(x - x_1)}{r_1^3} - \frac{\mu_2(x - x_2)}{r_2^3}$$

$$\ddot{y} = -2\omega\dot{x} + \omega^2 y - \dot{\omega} x - \frac{\mu_1 y}{r_1^3} - \frac{\mu_2 y}{r_2^3} \qquad (26)$$

$$\ddot{z} = -\frac{\mu_1 z}{r_1^3} - \frac{\mu_2 z}{r_2^3}$$

where

$$\omega^2(t) = \frac{\mu(t)}{\kappa r^3} \qquad (27)$$

Which are the equations of motion of the classical RTBP with variable masses of Gelf'gat (1973) and Bekov (1988).

The descriptions of the equations of motion of an infinitesimal mass in the gravitational environment of two massive bodies whose masses change with time and the bigger primary is an oblate spheroid whose oblate shape also varies with time with zonal harmonics up to J_4 term are obtained in this work. The equation of motion of the primaries is defined by the Gylden-Mestschersky problem (GMP), while the variations in the mass and oblateness of the bigger primary are described by UML. The oblate shape of the bigger primary varies with time due to its changing radius which is the coefficients of the zonal harmonics up to J_4 term. The equations of motion of the time varying dynamical system were derived using the Hamiltonian method. These equations are different from those of Gelf'gat (1973), Bekov (1988), Luk'yanov (1989), Singh and Leke (2010,2012), Gao et al (2024), Leke et al (2024&2025) due to the inclusion of the deforming shape of the bigger primary from a sphere to a oblate spheroids up to J_4 terms in the zonal harmonics coefficients. They are also different from those of Abouelmagd (2012) due to the variation in the masses and oblateness of the bigger primary. All previous studies of Singh and Leke (2010, 2012), Luk'yanov (1989) and Bekov (1988) be recovered from our equations.

The non-autonomous differential equations of the test particle in a rotating frame of reference are derived using the Hamiltonian method and these equations are described by the masses of the primaries, oblateness of the bigger primary and the angular velocity of revolution of the primaries. The RTBP is an active and stimulating research area that has been receiving attentions because of its applications to dynamics of small bodies in the solar and stellar systems and also because of its applications to satellite dynamics.

CONCLUSION

Hence, this paper contributes to knowledge by opening new problem involving mass variations and zonal harmonics which have remained open

The derivations of the dynamical equations will pave way for more explorations of the dynamical predictions of the test particle in the gravitational field of a binary system having the more massive body as an oblate spheroid under zonal harmonics coefficients up to J_4. This will in no doubt expand the knowledge base of celestial mechanics and space missions.

REFERENCE

AbdulRaheem, A. & Singh, J. (2006). Combined effects of radiation, perturbations and oblateness on locations and stability of equilibrium points of the restricted three-body problem. *Astron. J.* **131**, 1880

Abouelmagd, E. I. (2012). Existence and stability of triangular points in the restricted three-body problem with numerical applications. *AP&SS*, **342**,45-53.

Bekov, A. A. (1988). Libration points of the restricted problem of three bodies with variable mass. *Soviet Astronomy Journal*,**.33**, 92-95.

Bury, L. & McMahon, J., (2020). The effect of zonal harmonics on dynamical structures in the circular restricted three-body problem near the secondary body. *Celestial Mechanics and Dynamical Astronomy* **132**,45 .

Gao, F. Feng, Y. Wang, R. & Abouelmagd, E. I. (2024). Analysis of motion in RTBP with variable mass based on Loglogistic distribution. *Results in Physics*, **60,** 107637

Gelf'gat, B.E. (1973). *Current Problems of Celestial Mechanics and Astrodynamics*, Nauka, Moscow

Gyegwe, J.M, Omede. S., Ibrahim, M.A & Momoh, S.O (2025). The influence of J6 zonal harmonics on the location and stability of the CEPs of a Nigerian satellite in the generalized R3BP for EQ pegasi system. Journal of Basics and Applied Sciences Research, 3 (3) 261-276

Gylden, H. (1884). Die Bahnbewegungen in Einem Systeme von zwei Körpern in dem Falle, dass die Massen Ver Nderun- Gen Unterworfen Sind. *Astronomische Nachrichten*, **109**, 1-6.

Ibraimova, A. T., Minglibayev, M. Z. and Prokopenya, A.N. (2023). Perturbations in the restricted three-body problem of variable masses using computer algebra. *Computational Mathematics and Mathematical Physics*. **63**, 115-125.

Leke, O. & Mmaju, C. (2023). Zero Velocity Curves of a Dust Grain around Equilibrium Points Under Effects of Radiation, Perturbations and Variable Kruger 60. *Physics. Astronomy. International Journal* **7,** 280-285.

Leke, O. & Singh, J. (2023). Out-of-plane equilibrium points of extra-solar planets in the central binaries PSR b1620-26 and Kepler-16 with cluster of material points and variable masses. *New Astronomy*, **99**, 101958,

Leke, O. & Orum, S.A. (2024). Motion and Zero Velocity Curves of a Dust Grain Around Collinear Libration Points for the Binary IRAS 11472-0800 and G29-38 with a Triaxial Star and Variable Masses. *New Astronomy*, **108,** 102177

Leke, O. Cyril-Okeme, V. & Orum, S.A. (2024). Impact of Triaxiality and Mass Variations on Motion around Triangular Equilibrium Points of the Restricted Three-Body Problem, *Astronomy Reports*, **68**,1117-1141

Leke, O. Cyril-Okeme, V, Stephen, S, & Gyegwe, J. (2025). Investigation of motion around out-of-plane points in the restricted three-body problem with variable shape and masses. *New Astronomy*, **114,** 102311

Luk'yanov, L.G. (1989). Particular solutions in the restricted problem of three bodies with variable masses. *Astron. J. of Academy of Sciences of USSR*,.**66**, 180-18.

Mestschersky, I.V. (1902). Ueber die Integration der Bewegungs- gleichungen im Probleme zweier Körper von ver nderli- cher Masse. Astronomische Nachrichten. **159,** p229-242

Mestschersky, I.V. (1952). *Works on the Mechanics of Bodies of Variable Mass*. GITTL, Moscow, 205.

Singh, J. & Ishwar, B, V. (1999) Stability of triangular points in the generalized photogravitational restricted three-body problem. *Bulletin of the Astronomical Society of India*, **27,** 415-424.

Singh, J. &Leke, O. (2010) Stability of the photogravitational restricted three-body problem with variable masses. Astrophys. Space Sci. **326**, 305- 314 (2010).

Singh, J. & Leke, O. (2012). Equilibrium points and stability in the restricted three-body problem with oblateness and variable masses. Astrophys. Space Sci. **340,** 27-41.

Singh, J. & Leke, O. (2013). Effects of oblateness, perturbations, radiation and varying masses on the stability of equilibrium points in the restricted three-body problem. Astrophys. Space Sci. **344,** 51-61.

Singh, J. & Leke, O. (2014). Analytic and numerical treatment of motion of dust grain particle around triangular equilibrium points with post-AGB binary star and disc. *Advances in Space Research*, **54,** 1659–1677.

Singh, J. & Taura, J.J. (2015). Collinear Libration Points in the Photogravitational CR3BP with Zonal Harmonics and Potential from a Belt, *IJAA*, 5, 155-165

Szebehely, V.G. (1967). *Theory of Orbits*. Academic Press, New York.