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On the Modified Regression-Cum-Ratio Type Estimators of Population Variance in the presence of Non-Response and Measurement Error

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ABSTRACT

This study proposes three classes of modified regression-cum-ratio type estimators of the finite population variance under simple random sampling when both the study and auxiliary variables are affected by non-response and measurement errors. Each estimator combines a regression component with a dual-to-ratio, exponential, or ratio-exponential product transformation based on known auxiliary parameters. First-order expressions for the bias and mean squared error (MSE) were derived, together with conditions under which the proposed estimators are more efficient than existing ones. Performance was evaluated by Monte Carlo simulation on five artificial populations (N = 1,000, n = 100, 1,000 samples), with measurement error added to both variables. Using bias, MSE, and percentage relative efficiency (PRE), the estimators were compared with the sample variance estimator and the estimators of Isaki, Singh et al., Yadav and Kadilar, Asghar et al., and Audu et al. All proposed estimators had smaller MSEs than the sample variance estimator in every population. The exponential-type estimators Y22 and Y23 outperformed all existing estimators in every population, with Y22 attaining PREs of about 11,025%, 9,276%, 20,834%, 1,264%, and 20,833% in Populations I to V, respectively. The ratio-exponential product-type estimators were less efficient than some Audu et al. estimators in Populations II, III, and V. Biases of the proposed estimators were small, except in Population IV, where all estimators were markedly biased. The proposed estimators are therefore efficient alternatives for variance estimation under non-response and measurement error.

Keywords:

Population variance,
Non-response,
Measurement error,
Auxiliary information,
Regression-cum-ratio estimator.

INTRODUCTION

Population variance is one of the parameters which require efficient estimators because of its great significance in various fields such as in matters of health: variations in body temperature, pulse beat and blood pressure which are the basic guides of diagnosis when prescribed treatment is designed to control their variation. The variation of produce or yields in the manufacturing industries and pharmaceutical laboratories are sometime a matter of concern to researchers (Ahmed et al. (2003)). The population variance may also play significant role in the study of variations of produce or yields in the manufacturing industries and pharmaceutical laboratories. Many other researchers including Isaki (1983), Singh and Singh (2001), Singh and Singh (2003), Kadilar and Cingi (2006), Yadav and Kadilar (2013), Singh and Solanki (2013), Gupta and Shabir (2008), Subramani and Kumarapandiyan (2014), Singh and Vishwakarma (2008), Singh et al. (2007), Singh et al. (2011), Sanaullah et al. (2012), Diana and Giordan (2012), Solanki and Singh (2013), Ahmed et al. (2016b,c,d) have significantly contributed to the improvement of both ratio and product mean & variance estimators in sampling survey.

Statistically, it is usually presumed that all the observations are measured correctly. However, in practical situations this assumption is somewhat violated and the true value of the variable cannot be recorded. In such cases there is always some

associated error with the variable that leads to deviation between true and observed values of the variable. Such deviations are termed as measurement error. Several authors including Shalabh (1997), Singh and Karpe (2009), Manisha and Singh (2001), Kumar et. al (2011), Sharma and Singh (2013), Malik and Singh (2013), Singh and Sharma (2015), Azeem and Hanif (2015), Singh et al. (2019) have considered estimation in the presence of measurement error. Another type of non-sampling error that generally creeps in sampling surveys includes non-response error that is unavoidable. Hansen and Hurwitz (1946) were the first to deal with the problem of non-response by using technique of subsampling from non-respondents. Non-response may generally arise due to non-availability of respondents, refusal to respond, presence of hardcore respondents or due to non-understanding of the particular question. The estimation procedures under non-response are considered in literature by many authors for instance Khare and Srivastava (1993,1997), Yaqub et. al (2017), etc. In Survey Sampling, situation may arise that both of these errors occur simultaneously. Similarly, studies on the estimation of population mean in the presence of simultaneous effects of measurement and non-response errors have been considered by authors namely Jackmann (1999), Dixon (2010). Kumar et. al (2015), Poonam et al. (2018), Singh et al. (2018), Audu et al. (2021) and Audu et al. (2022).

Reviewed of Some Related Existing Estimators

Usual variance estimator when study variable is characterized by non-response and measurement error is given as;

$$s^{*2}_{y(e)} = n_1 n^{-1} s^2_{y1(e)} + n_2 n^{-1} s^2_{yh_2(e)} = \hat{t}^*_{0e}, (say) \quad (1)$$

$$Bias(\hat{t}^*_{0e}) = 0 \quad (2)$$

$$MSE(\hat{t}^*_{0e}) = \sigma^4_Y\left(k_1 H_Y + k_2 H_{Y(h_2)}\right) = \sigma^4_Y H_0, (say) \quad (3)$$

where $H_0 = k_1 H_Y + k_2 H_{Y(h_2)}, H_Y = \lambda_{40} + \gamma_{40}\sigma^4_U\sigma_Y^{-4} + 2(\sigma^2_U\sigma_Y^{-2})^2$,
$H_{Y(h_2)} = \lambda_{40(h_2)} + \gamma_{40(h_2)}\sigma^4_{U(h_2)}\sigma^{-4}_{Y(h_2)} + 2\left(1 + \sigma^2_{U(h_2)}\sigma^{-2}_{Y(h_2)}\right)^2$,
$k_1 = n^{-1} - N^{-1}, k_2 = W_2(f-1)n^{-1}$

Isaki (1983) variance estimator under the influences of non-response and measurement error can be defined as;

$$\hat{t}^*_1 = s^{*2}_{y(e)} S^2_X s^{*-2}_{x(e)} \quad (4)$$

$$Bias(\hat{t}^*_1) = \sigma^4_Y(H_1 - H_{01}) \quad (5)$$

$$MSE(\hat{t}^*_1) = \sigma^4_Y(H_0 + H_1 - 2H_{02}) \quad (6)$$

where $s^{*2}_{y(e)} = n_1 n^{-1} s^2_{x1(e)} + n_2 n^{-1} s^2_{xh_2(e)}$, $H_2 = k_1 H_X + k_2 H_{X(h_2)}, H_X = \lambda_{04} + \gamma_{04}\sigma^4_V\sigma_X^{-4} + 2(1 + \sigma^2_V\sigma_X^{-2})^2$,
$H_{Y(h_2)} = \lambda_{04(h_2)} + \gamma_{04(h_2)}\sigma^4_{V(h_2)}\sigma^{-4}_{X(h_2)} + 2\left(1 + \sigma^2_{V(h_2)}\sigma^{-2}_{X(h_2)}\right)^2$, $H_{02} = k_1\lambda_{22} + k_2\lambda_{22(2)}$,

Singh et al. (2011) estimator when both study and auxiliary variables are characterized by non-response and measurement error is defined as;

$$\hat{t}^*_2 = s^{*2}_{ye} \exp\left(\frac{S^2_X - s^{*2}_{xe}}{S^2_X + s^{*2}_{xe}}\right) \quad (7)$$

$$Bias(\hat{t}^*_2) = \sigma^2_Y\left(\frac{3}{8}H_1 - \frac{1}{2}H_{01}\right) \quad (8)$$

$$MSE(\hat{t}^*_2) = \sigma^4_Y(H_0 + 0.25H_1 - H_{01}) \quad (9)$$

Yadav and Kadilar (2013) variance estimator under simultaneous effects of non-response and measurement error on both the study and auxiliary characters is defined as;

$$\hat{t}^*_3 = s^{*2}_{ye} \exp\left(\frac{S^2_X - s^{*2}_{xe}}{S^2_X + (1-\alpha)s^{*2}_{xe}}\right) \quad (10)$$

$$Bias(\hat{t}^*_3) = \frac{\sigma^2_Y\left((2\alpha - 1)H_1 - H_{01}\right)}{2\alpha^2} \quad (11)$$

$$MSE(\hat{t}^*_3) = \sigma^4_Y\left(H_0 - \frac{H^2_{01}}{H_1}\right) \quad (12)$$

Asghar et al. (2014) suggested ratio exponential-type estimator. This estimator, when both study and auxiliary variables are characterized by non-response and measurement error is defined as;

$$\hat{t}^*_3 = s^{*2}_{ye} \exp\left(\frac{\bar{X} - \bar{x}^*_e}{\bar{X} + \bar{x}^*_e}\right) \quad (13)$$

$$Bias(\hat{t}^*_4) = \sigma^2_Y\left(\frac{3}{8}H_2 - \frac{1}{2}H_{02}\right) \quad (14)$$

$$MSE(\hat{t}^*_4) = \sigma^4_Y(H_0 + 0.25H_2 - H_{02}) \quad (15)$$

Where $H_1 = k_1 C^2_X(1 + \sigma^2_V\sigma_X^{-2}) + k_2 C^2_{X(2)}\left(1 + \sigma^2_V\sigma^{-2}_{X(2)}\right), H_{01} = k_1\delta_{12}C_X + k_2\delta_{12(2)}C_{X(2)}$

Audu et al. (2022) suggested three classes of almost unbiased estimators of population variance as follow;

$$\hat{t}^*_{i1} = \sum_{j=0}^{2} 3\omega_{ij}\hat{q}^*_{ij}, \quad \forall\, i = 1,2 \quad (16)$$

$$\hat{t}^*_{i2} = \sum_{j=0}^{2} v_{ij} z^*_{ij}, \quad \forall\, i = 1,2 \quad (17)$$

$$\hat{t}^*_{i3} = \sum_{j=0}^{2} \tau_{ij} p^*_{ij}, \quad \forall\, i = 1,2 \quad (18)$$

where

$$q_{10}^{*} = s_{y(e)}^{*2}, q_{11}^{*} = s_{y(e)}^{*2}(A_X\mu_X + B_X)(A_X\bar{x}_e^{*} + B_X)^{-1}, q_{12}^{*} = s_{y(e)}^{*2}(A_X\mu_X + B_X)^{-1}(A_X\bar{x}_e^{*} + B_X)$$

$$z_{10}^{*} = q_{10}^{*}, z_{11}^{*} = q_{11}^{*}, z_{12}^{*} = z_{10}^{*}\exp\left(\left((A_X\bar{x}_e^{*} + B_X) - (A_X\mu_X + B_X)\right)\left((A_X\bar{x}_e^{*} + B_X) + (A_X\mu_X + B_X)\right)^{-1}\right)$$

$$p_{10}^{*} = z_{10}^{*}, p_{11}^{*} = p_{10}^{*}\exp\left(\left((A_X\bar{x}_e^{*} + B_X) - (A_X\mu_X + B_X)\right)^{-1}\left((A_X\bar{x}_e^{*} + B_X) + (A_X\mu_X + B_X)\right)\right), p_{12}^{*} = z_{12}^{*}$$

$$q_{20}^{*} = q_{10}^{*}, q_{21}^{*} = q_{20}^{*}(A_XS_X^2 + B_X)\left(A_Xs_{x(e)}^{*2} + B_X\right)^{-1}, q_{22}^{*} = q_{20}^{*}(A_XS_X^2 + B_X)^{-1}\left(A_Xs_{x(e)}^{*2} + B_X\right)$$

$$z_{20}^{*} = p_{20}^{*}, z_{21}^{*} = q_{21}^{*}, z_{22}^{*} = z_{20}^{*}\exp\left(\left(\left(A_Xs_{x(e)}^{*2} + B_X\right) - (A_XS_X^2 + B_X)\right)\left(\left(A_Xs_{x(e)}^{*2} + B_X\right) + (A_XS_X^2 + B_X)\right)^{-1}\right)$$

$$l_0^{*} = \mu_0^{*}, p_{21}^{*} = z^{*}, p_{22}^{*} = p_{20}^{*}\exp\left(\left(\left(A_Xs_{x(e)}^{*2} + B_X\right) - (A_XS_X^2 + B_X)\right)^{-1}\left(\left(A_Xs_{x(e)}^{*2} + B_X\right) + (A_XS_X^2 + B_X)\right)\right),$$

$$\sum_{j=0}^{2}\omega_{ij} = \sum_{j=0}^{2}v_{ij} = \sum_{j=0}^{2}\tau_{ij} = 1, i = 1,2$$

$$Bias\,(\hat{t}_{i1}^{*}) = \sigma_Y^2(\omega_{i1}\psi_i^2\mathrm{H_i} - (\omega_{i1} - \omega_{i2})\psi_iH_{0i}) \quad (19)$$

$$Bias(\hat{t}_{i2}^{*}) = \sigma_Y^2\left(\left(v_{i1} - \frac{v_{i2}}{8}\right)\psi_i^2H_i - \left(v_{i1} - \frac{v_{i2}}{2}\right)\psi_iH_{oi}\right) \quad (20)$$

$$Bias(\hat{t}_{i3}^{*}) = S_Y^2(8^{-1}(3\tau_{i1} - \tau_{i2})\psi_i^2H_i - 2^{-1}(\tau_{i1} - \tau_{i2})\psi_iH_{0i}) \quad (21)$$

$$MSE(\hat{t}_{i1}^{*}) = \sigma_Y^4(\mathrm{H_0} + \theta_i^2\psi_i^2\mathrm{H_i} - 2\theta_i\psi_iH_{0i}) \quad (22)$$

$$MSE(\hat{t}_{i2}^{*}) = \sigma_Y^4(\mathrm{H_0} + \vartheta_i^2\psi_i^2\mathrm{H_i} - 2\vartheta_i\psi_iH_{0i}) \quad (23)$$

$$MSE(\hat{t}_{i3}^{*}) = \sigma_Y^4(\mathrm{H_0} + 2^{-2}\phi_i^2\psi_i^2\mathrm{H_i} - 2\phi_i\psi_iH_{0i}) \quad (24)$$

where

$$\omega_{i0} = 1 + \theta_i^{(opt)} - 2\theta_i^{(opt)2}, \omega_{i1} = \theta_i^{(opt)2}, \omega_{i2} = -\theta_i^{(opt)} + \theta_i^{(opt)2}$$

$$v_{i0} = 1 + 3\vartheta_i^{(opt)} - 4\theta_i^{(opt)2}, v_{i1} = -\vartheta_i^{(opt)}\left(1 - 4\vartheta_i^{(opt)}\right)/3, v_{i2} = -8\vartheta_i^{(opt)}\left(1 - \vartheta_i^{(opt)}\right)/3$$

$$\tau_{i0} = 1 + 4\phi_i^{(opt)} - 8\phi_i^{(opt)2}, \tau_{i1} = -\phi_i^{(opt)}\left(1 - 4\phi_i^{(opt)}\right), \tau_{i2} = -\phi_i^{(opt)}\left(3 - 4\phi_i^{(opt)}\right)$$

$$\theta_i = \omega_{i1} - \omega_{i2}, \vartheta_i = (v_{i1} - v_{i2}/2), \phi_i = \tau_{i1} - \tau_{i2},$$

$$\psi_1 = A_X\bar{X}/(A_X\bar{X} + B_X),\ \psi_2 = A_XS_X^2/(A_XS_X^2 + B_X)$$

3.0 Proposed Estimators for Population Variance in the Presence of Non-Response and Measurement Errors

Having studied the work of Audu et al. (2022) and adopting the concept of dual-to-ratio estimator concept as presented in the first seminar, the following modified classes of estimators of population variance in the presence of non-response and measurement error are proposed;

$$\Upsilon_{1j} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{(A_X\,(NS_X^2 - ns_{xe}^{*2})/(N-n) + B_X)}(A_XS_X^2 + B_X) \quad (25)$$

$$\Upsilon_{2j} = \left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)\exp\left[\frac{(A_XS_X^2 + B_X) - (A_X\,(NS_X^2 - ns_{xe}^{*2})/(N-n) + B_X)}{(A_XS_X^2 + B_X) + (A_X\,(NS_X^2 - ns_{xe}^{*2})/(N-n) + B_X)}\right] \quad (26)$$

$$\Upsilon_{3j} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{(A_X\,(NS_X^2 - ns_{xe}^{*2})/(N-n) + B_X)}(A_XS_X^2 + B_X) \times \exp\left[\frac{(A_XS_X^2 + B_X) - (A_X\,(NS_X^2 - ns_{xe}^{*2})/(N-n) + B_X)}{(A_XS_X^2 + B_X) + (A_X\,(NS_X^2 - ns_{xe}^{*2})/(N-n) + B_X)}\right] \quad (27)$$

Where A_X, B_X are available auxiliary information like coefficient of skewness, kurtosis, variation, and standard deviation.

Remark 3.1: Note that $A_X \neq B_X$ and $A_X \neq 0$.

3.1 Members of the Proposed Estimators

In this subsection, members of the classes of the proposed are presented in the tables below.

Table 1: Members of the Proposed Estimators $\Upsilon_{1j}, j = 1,2,3,\ldots,17$

S/N	A_X	B_X	Members of Proposed Estimator Υ_{1j}
1	1	0	$\Upsilon_{11} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{((NS_X^2 - ns_{xe}^{*2})/(N-n))}S_X^2$
2	1	S_X	$\Upsilon_{12} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{((NS_X^2 - ns_{xe}^{*2})/(N-n) + S_X)}(S_X^2 + S_X)$
3	1	C_X	$\Upsilon_{13} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{((NS_X^2 - ns_{xe}^{*2})/(N-n) + C_X)}(S_X^2 + C_X)$

4	**1**	$\beta_1(X$	$\Upsilon_{14} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{\left((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_1(X)\right)}\left(S_X^2 + \beta_1(X)\right)$
5	**1**	$\beta_2(X$	$\Upsilon_{14} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{\left((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_2(X)\right)}\left(S_X^2 + \beta_2(X)\right)$
6	S_X	C_X	$\Upsilon_{16} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{\left(S_X\,(NS_X^2 - ns_{xe}^{*2})/(N-n) + C_X\right)}(S_X^3 + C_X)$
7	S_X	$\beta_1(X$	$\Upsilon_{17} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{\left(S_X\,(NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_1(X)\right)}(S_X^3 + \beta_1(X))$
8	S_X	$\beta_2(X$	$\Upsilon_{18} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{\left(S_X\,(NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_2(X)\right)}(S_X^3 + \beta_2(X))$
9	C_X	S_X	$\Upsilon_{19} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{\left(C_X\,(NS_X^2 - ns_{xe}^{*2})/(N-n) + S_X\right)}(C_X S_X^2 + S_X)$
10	C_X	$\beta_1(X$	$\Upsilon_{1(10)} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{\left(C_X\,(NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_1(X)\right)}(C_X S_X^2 + \beta_1(X))$
11	C_X	$\beta_2(X$	$\Upsilon_{1(11)} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{\left(C_X\,(NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_2(X)\right)}(C_X S_X^2 + \beta_2(X))$
12	$\beta_1(X$	S_X	$\Upsilon_{1(12)} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{\left(\beta_1(X)\,(NS_X^2 - ns_{xe}^{*2})/(N-n) + S_X\right)}(\beta_1(X) S_X^2 + S_X)$
13	$\beta_1(X$	C_X	$\Upsilon_{1(13)} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{\left(\beta_2(X)\,(NS_X^2 - ns_{xe}^{*2})/(N-n) + C_X\right)}(\beta_2(X) S_X^2 + C_X)$
14	$\beta_1(X$	$\beta_2(X$	$\Upsilon_{1(14)} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{\left(\beta_1(X)\,(NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_2(X)\right)}(\beta_1(X) S_X^2 + \beta_2(X))$
15	$\beta_2(X$	S_X	$\Upsilon_{1(15)} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{\left(\beta_2(X)\,(NS_X^2 - ns_{xe}^{*2})/(N-n) + S_X\right)}(\beta_2(X) S_X^2 + S_X)$
16	$\beta_2(X$	C_X	$\Upsilon_{1(16)} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{\left(\beta_2(X)\,(NS_X^2 - ns_{xe}^{*2})/(N-n) + C_X\right)}(\beta_2(X) S_X^2 + C_X)$
17	$\beta_2(X$	$\beta_1(X$	$\Upsilon_{1(17)} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{\left(\beta_2(X)\,(NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_1(X)\right)}(\beta_2(X) S_X^2 + \beta_1(X))$

Table 2: Members of the Proposed Estimators $\Upsilon_{2j}, j = 1, 2, 3, \ldots, 17$

S/N	A_X	B_X	Members of Proposed Estimator T_{2j}
1	**1**	**0**	$\Upsilon_{21} = \left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)\exp\left[\frac{S_X^2 - ((NS_X^2 - ns_{xe}^{*2})/(N-n))}{S_X^2 + ((NS_X^2 - ns_{xe}^{*2})/(N-n))}\right]$
2	**1**	S_X	$\Upsilon_{22} = \left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)\exp\left[\frac{(S_X^2 + S_X) - ((NS_X^2 - ns_{xe}^{*2})/(N-n) + S_X)}{(S_X^2 + S_X) + ((NS_X^2 - ns_{xe}^{*2})/(N-n) + S_X)}\right]$
3	**1**	C_X	$\Upsilon_{23} = \left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)\exp\left[\frac{(S_X^2 + C_X) - ((NS_X^2 - ns_{xe}^{*2})/(N-n) + C_X)}{(S_X^2 + C_X) + ((NS_X^2 - ns_{xe}^{*2})/(N-n) + C_X)}\right]$
4	**1**	$\beta_1(X)$	$\Upsilon_{24} = \left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)\exp\left[\frac{(S_X^2 + \beta_1(X)) - ((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_1(X))}{(S_X^2 + \beta_1(X)) + ((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_1(X))}\right]$
5	**1**	$\beta_2(X)$	$\Upsilon_{25} = \left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)\exp\left[\frac{(S_X^2 + \beta_2(X)) - ((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_2(X))}{(S_X^2 + \beta_2(X)) + ((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_2(X))}\right]$
6	S_X	C_X	$\Upsilon_{26} = \left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)\exp\left[\frac{(S_X^3 + C_X) - ((NS_X^2 - ns_{xe}^{*2})/(N-n) + C_X)}{(S_X^3 + C_X) + ((NS_X^2 - ns_{xe}^{*2})/(N-n) + C_X)}\right]$
7	S_X	$\beta_1(X)$	$\Upsilon_{27} = \left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)\exp\left[\frac{(S_X^3 + \beta_1(X)) - ((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_1(X))}{(S_X^3 + \beta_1(X)) + ((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_1(X))}\right]$

8	S_X	$\beta_2(X)$	$\Upsilon_{28} = \left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right) \exp\left[\frac{(S_X^3 + \beta_2(X)) - ((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_2(X))}{(S_X^3 + \beta_2(X)) + ((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_2(X))}\right]$
9	C_X	S_X	$\Upsilon_{29} = \left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right) \exp\left[\frac{(C_X S_X^2 + S_X) - ((NS_X^2 - ns_{xe}^{*2})/(N-n) + S_X)}{(C_X S_X^2 + S_X) + ((NS_X^2 - ns_{xe}^{*2})/(N-n) + S_X)}\right]$
10	C_X	$\beta_1(X)$	$\Upsilon_{2(10)} = \left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right) \exp\left[\frac{(C_X S_X^2 + \beta_1(X)) - ((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_1(X))}{(C_X S_X^2 + \beta_1(X)) + ((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_1(X))}\right]$
11	C_X	$\beta_2(X)$	$\Upsilon_{2(11)} = \left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right) \exp\left[\frac{(C_X S_X^2 + \beta_2(X)) - ((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_2(X))}{(C_X S_X^2 + \beta_2(X)) + ((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_2(X))}\right]$
12	$\beta_1(X)$	S_X	$\Upsilon_{2(12)} = \left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right) \exp\left[\frac{(\beta_1(X) S_X^2 + S_X) - ((NS_X^2 - ns_{xe}^{*2})/(N-n) + S_X)}{(\beta_1(X) S_X^2 + S_X) + ((NS_X^2 - ns_{xe}^{*2})/(N-n) + S_X)}\right]$
13	$\beta_1(X)$	C_X	$\Upsilon_{2(13)} = \left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right) \exp\left[\frac{(\beta_1(X) S_X^2 + C_X) - ((NS_X^2 - ns_{xe}^{*2})/(N-n) + C_X)}{(\beta_1(X) S_X^2 + C_X) + ((NS_X^2 - ns_{xe}^{*2})/(N-n) + C_X)}\right]$
14	$\beta_1(X)$	$\beta_2(X)$	$\Upsilon_{2(14)} = \left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right) \exp\left[\frac{(\beta_1(X) S_X^2 + B_2(X)) - ((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_2(X))}{(\beta_1(X) S_X^2 + B_2(X)) + ((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_2(X))}\right]$
15	$\beta_2(X)$	S_X	$\Upsilon_{2(15)} = \left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right) \exp\left[\frac{(\beta_2(X) S_X^2 + S_X) - ((NS_X^2 - ns_{xe}^{*2})/(N-n) + S_X)}{(\beta_2(X) S_X^2 + S_X) + ((NS_X^2 - ns_{xe}^{*2})/(N-n) + S_X)}\right]$
16	$\beta_2(X)$	C_X	$\Upsilon_{2(16)} = \left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right) \exp\left[\frac{(\beta_2(X) S_X^2 + C_X) - ((NS_X^2 - ns_{xe}^{*2})/(N-n) + C_X)}{(\beta_2(X) S_X^2 + C_X) + ((NS_X^2 - ns_{xe}^{*2})/(N-n) + C_X)}\right]$
17	$\beta_2(X)$	$\beta_1(X)$	$\Upsilon_{2(15)} = \left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right) \exp\left[\frac{(\beta_2(X) S_X^2 + \beta_1(X)) - ((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_1(X))}{(\beta_2(X) S_X^2 + \beta_1(X)) + ((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_1(X))}\right]$

Table 3: Members of the Proposed Estimators $T_{3j}, j = 1,2,3,\ldots,17$

S/N	A_X	B_X	Members of Proposed Estimator Υ_{3j}
1	**1**	**0**	$\Upsilon_{31} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{((NS_X^2 - ns_{xe}^{*2})/(N-n))} S_X^2 \times \exp\left[\frac{S_X^2 - ((NS_X^2 - ns_{xe}^{*2})/(N-n))}{S_X^2 + ((NS_X^2 - ns_{xe}^{*2})/(N-n))}\right]$
2	**1**	S_X	$\Upsilon_{32} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{((NS_X^2 - ns_{xe}^{*2})/(N-n) + S_X)} (S_X^2 + S_X) \times \exp\left[\frac{(S_X^2 + S_X) - ((NS_X^2 - ns_{xe}^{*2})/(N-n) + S_X)}{(S_X^2 + S_X) + ((NS_X^2 - ns_{xe}^{*2})/(N-n) + S_X)}\right]$
3	**1**	C_X	$\Upsilon_{32} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{((NS_X^2 - ns_{xe}^{*2})/(N-n) + C_X)} (S_X^2 + C_X) \times \exp\left[\frac{(S_X^2 + C_X) - ((NS_X^2 - ns_{xe}^{*2})/(N-n) + C_X)}{(S_X^2 + C_X) + ((NS_X^2 - ns_{xe}^{*2})/(N-n) + C_X)}\right]$
4	**1**	$\beta_1(X)$	$\Upsilon_{34} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{((NS_X^2 - ns_{xe}^{*2})/(N-n) + B_1(X))} \left(S_X^2 + \beta_1(X)\right) \times \exp\left[\frac{\left(S_X^2 + \beta_1(X)\right) - ((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_1(X))}{\left(S_X^2 + \beta_1(X)\right) + \left((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_1(X)\right)}\right]$
5	**1**	$\beta_2(X)$	$\Upsilon_{35} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{((NS_X^2 - ns_{xe}^{*2})/(N-n) + B_2(X))} \left(S_X^2 + \beta_2(X)\right) \times \exp\left[\frac{\left(S_X^2 + \beta_2(X)\right) - ((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_2(X))}{\left(S_X^2 + \beta_2(X)\right) + \left((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_2(X)\right)}\right]$
6	S_X	C_X	$\Upsilon_{36} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{((NS_X^2 - ns_{xe}^{*2})/(N-n) + C_X)} (S_X^3 + C_X) \times \exp\left[\frac{(S_X^3 + C_X) - ((NS_X^2 - ns_{xe}^{*2})/(N-n) + C_X)}{(S_X^3 + C_X) + ((NS_X^2 - ns_{xe}^{*2})/(N-n) + C_X)}\right]$

7	S_X	$\beta_1(X)$	$\Upsilon_{37} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{((NS_X^2 - ns_{xe}^{*2})/(N-n) + B_1(X))}\left(S_X^3 + \beta_1(X)\right) \times \exp\left[\frac{\left(S_X^3 + \beta_1(X)\right) - ((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_1(X))}{\left(S_X^3 + \beta_1(X)\right) + ((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_1(X))}\right]$
8	S_X	$\beta_2(X)$	$\Upsilon_{38} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{((NS_X^2 - ns_{xe}^{*2})/(N-n) + B_2(X))}\left(S_X^3 + \beta_2(X)\right) \times \exp\left[\frac{\left(S_X^3 + \beta_2(X)\right) - ((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_2(X))}{\left(S_X^3 + \beta_2(X)\right) + ((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_2(X))}\right]$
9	C_X	S_X	$\Upsilon_{39} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{((NS_X^2 - ns_{xe}^{*2})/(N-n) + S_X)}(C_X S_X^2 + S_X) \times \exp\left[\frac{(C_X S_X^2 + S_X) - ((NS_X^2 - ns_{xe}^{*2})/(N-n) + S_X)}{(C_X S_X^2 + S_X) + ((NS_X^2 - ns_{xe}^{*2})/(N-n) + S_X)}\right]$
10	C_X	$\beta_1(X)$	$\Upsilon_{3(10)} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_1(X))}\left(C_X S_X^2 + \beta_1(X)\right) \times \exp\left[\frac{\left(C_X S_X^2 + \beta_1(X)\right) - ((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_1(X))}{\left(C_X S_X^2 + \beta_1(X)\right) + ((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_1(X))}\right]$
11	C_X	$\beta_2(X)$	$\Upsilon_{3(11)} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_2(X))}\left(C_X S_X^2 + \beta_2(X)\right) \times \exp\left[\frac{\left(C_X S_X^2 + \beta_2(X)\right) - ((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_2(X))}{\left(C_X S_X^2 + \beta_2(X)\right) + ((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_2(X))}\right]$
12	$\beta_1(X)$	S_X	$\Upsilon_{3(12)} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{((NS_X^2 - ns_{xe}^{*2})/(N-n) + S_X)}(\beta_1(X) S_X^2 + S_X) \times \exp\left[\frac{(\beta_1(X) S_X^2 + S_X) - ((NS_X^2 - ns_{xe}^{*2})/(N-n) + S_X)}{(\beta_1(X) S_X^2 + S_X) + ((NS_X^2 - ns_{xe}^{*2})/(N-n) + S_X)}\right]$
13	$\beta_1(X)$	C_X	$\Upsilon_{3(13)} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{((NS_X^2 - ns_{xe}^{*2})/(N-n) + C_X)}(\beta_1(X) S_X^2 + C_X) \times \exp\left[\frac{(\beta_1(X) S_X^2 + C_X) - ((NS_X^2 - ns_{xe}^{*2})/(N-n) + C_X)}{(\beta_1(X) S_X^2 + C_X) + ((NS_X^2 - ns_{xe}^{*2})/(N-n) + C_X)}\right]$
14	$\beta_1(X)$	$\beta_2(X)$	$\Upsilon_{3(14)} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_2(X))}\left(\beta_1(X) S_X^2 + \beta_2(X)\right) \times \exp\left[\frac{\left(\beta_1(X) S_X^2 + \beta_2(X)\right) - ((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_2(X))}{\left(\beta_1(X) S_X^2 + \beta_2(X)\right) + ((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_2(X))}\right]$
15	$\beta_2(X)$	S_X	$\Upsilon_{3(15)} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{((NS_X^2 - ns_{xe}^{*2})/(N-n) + S_X)}(\beta_2(X) S_X^2 + S_X) \times \exp\left[\frac{(\beta_2(X) S_X^2 + S_X) - ((NS_X^2 - ns_{xe}^{*2})/(N-n) + S_X)}{(\beta_2(X) S_X^2 + S_X) + ((NS_X^2 - ns_{xe}^{*2})/(N-n) + S_X)}\right]$
16	$\beta_2(X)$	C_X	$\Upsilon_{3(16)} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{((NS_X^2 - ns_{xe}^{*2})/(N-n) + C_X)}(\beta_2(X) S_X^2 + C_X) \times \exp\left[\frac{(\beta_2(X) S_X^2 + C_X) - ((NS_X^2 - ns_{xe}^{*2})/(N-n) + C_X)}{(\beta_2(X) S_X^2 + C_X) + ((NS_X^2 - ns_{xe}^{*2})/(N-n) + C_X)}\right]$
17	$\beta_2(X)$	$\beta_1(X)$	$\Upsilon_{3(17)} = \frac{\left(s_{ye}^{*2} + b_{rg}(S_X^2 - s_{xe}^{*2})\right)}{((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_1(X))}\left(\beta_2(X) S_X^2 + \beta_1(X)\right) \times \exp\left[\frac{\left(\beta_2(X) S_X^2 + \beta_1(X)\right) - ((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_1(X))}{\left(\beta_2(X) S_X^2 + \beta_1(X)\right) + ((NS_X^2 - ns_{xe}^{*2})/(N-n) + \beta_1(X))}\right]$

3.3 Bias and Mean Square Error of Proposed Estimators

This subsection presents the biases and MSEs of the Proposed Estimators of Population Variance. To obtain the biases and MSEs of the proposed estimators of population variance, the following error terms $\varepsilon_0^* = (s_{ye}^{*2} - \sigma_Y^2)/\sigma_Y^2, \varepsilon_1^* = (s_{xe}^{*2} - \sigma_X^2)/\sigma_X^2$, such that $\left|\varepsilon_i^*\right| \approx 0,\ i = 0,1.$ From the definitions of the error terms, $s_{ye}^{*2} = (1 + \varepsilon_0^*)\sigma_Y^2, s_{xe}^{*2} = (1 + \varepsilon_1^*)\sigma_X^2$. The expected values of ε_i^* up to $O(n^{-1})$ are given as;

$$\left.\begin{aligned} &E(\varepsilon_0^*) = E(\varepsilon_1^*) = 0, E(\varepsilon_0^{*2}) = H_0, \\ &E(\varepsilon_1^{*2}) = H_2, \quad E(\varepsilon_0^*\varepsilon_1^*) = H_{02} \end{aligned}\right\} \quad (28)$$

where $H_0 = k_1 H_Y + k_2 H_{Y(h_2)},\ H_Y = \lambda_{40} + \gamma_{40}\sigma_U^4\sigma_Y^{-4} + 2(1 + \sigma_U^2\sigma_Y^{-2})^2,$

$H_{Y(h_2)} = \lambda_{40(h_2)} + \gamma_{40(h_2)}\sigma_{U(h_2)}^4\sigma_{Y(h_2)}^{-4} + 2\left(1 + \sigma_{U(h_2)}^2\sigma_{Y(h_2)}^{-2}\right)^2, k_1 = n^{-1} - N^{-1},$

$k_2 = W_2(f-1)n^{-1},\ H_1 = k_1 H_X + k_2 H_{X(h_2)}, H_X = \lambda_{04} + \gamma_{04}\sigma_V^4\sigma_X^{-4} + 2(1 + \sigma_V^2\sigma_X^{-2})^2,$

$H_{01} = k_1\lambda_{22} + k_2\lambda_{22(2)}$

The estimators $\Upsilon_{1j}, j = 1,2,3,\ldots,17$ can be expressed in terms of errors $\varepsilon_0^*, \varepsilon_1^*$as

$$\Upsilon_{1j} = \frac{\left((1 + \varepsilon_0^*)\sigma_Y^2 + b_{rg}(S_X^2 - (1 + \varepsilon_1^*)\sigma_X^2)\right)}{(A_X\,(NS_X^2 - n(1 + \varepsilon_1^*)\sigma_Y^2)/(N-n) + B_X)}(A_X S_X^2 + B_X) \quad (29)$$

As $N \to \infty$, $S_X^2 \approx \sigma_X^2$ then equation (29) becomes (30).

$$\Upsilon_{1j} = \frac{\left(\sigma_Y^2 + \sigma_Y^2\varepsilon_0^* - b_{rg}\sigma_X^2\varepsilon_1^*\right)}{(A_X\,(\sigma_X^2(N-n) - n\sigma_X^2\varepsilon_1^*)/(N-n) + B_X)}(A_X\sigma_X^2 + B_X) \quad (30)$$

By simplifying (30), (31) is obtained

$$\Upsilon_{1j} = \left(\sigma_Y^2 + \sigma_Y^2\varepsilon_0^* - b_{rg}\sigma_X^2\varepsilon_1^*\right)(1 - n\lambda_k\sigma_X^2\varepsilon_1^*/(N-n))^{-1}$$

Where $\lambda_k = A_X\sigma_X^2/(A_X\sigma_X^2 + B_X)$

Simplify (31) to first order approximation, (32) is obtained

$$\Upsilon_{1j} - \sigma_Y^2 = \sigma_Y^2\varepsilon_0^* + \left(\frac{n\lambda_k\sigma_Y^2}{N-n} - b_{rg}\sigma_X^2\right)\varepsilon_1^* + \frac{n\lambda_k}{N-n}\left(\frac{n\lambda_k\sigma_Y^2}{N-n} - b_{rg}\sigma_X^2\right)\varepsilon_1^{*2} + \frac{n\lambda_k\sigma_Y^2}{N-n}\varepsilon_0^*\varepsilon_1^* \quad (32)$$

By taking expectation of both sides of (32), the bias of $\Upsilon_{1j}, j = 1,2,\ldots,17$ is obtained as

$$Bias(\Upsilon_{1j}) = \frac{n\lambda_k}{N-n}\left(\left(\frac{n\lambda_k\sigma_Y^2}{N-n} - b_{rg}\sigma_X^2\right)H_1 + \sigma_Y^2 H_{01}\right), \quad j = 1,2,\ldots,17 \quad (33)$$

Similarly, by squaring (32) and take expectation of the results, the MSE of $\Upsilon_{1j},\ j = 1,2,\ldots,17$ is obtained as

$$MSE(\Upsilon_{1j}) = \sigma_Y^2 H_0 + \left(\frac{n\lambda_k\sigma_Y^2}{N-n} - b_{rg}\sigma_X^2\right)^2 H_1 + 2\sigma_Y^2\left(\frac{n\lambda_k\sigma_Y^2}{N-n} - b_{rg}\sigma_X^2\right)H_{01}, j = 1,2,\ldots,17 \quad (34)$$

The estimators $\Upsilon_{2j}, j = 1,2,\ldots,17$ can be expressed in terms of errors as

$$\Upsilon_{2j} = \begin{pmatrix}(1 + \varepsilon_0^*)\sigma_Y^2 \\ + b_{rg}(S_X^2 - (1 + \varepsilon_1^*)\sigma_X^2)\end{pmatrix} \exp\left(\frac{(A_X S_X^2 + B_X) - \left(A_X\left(NS_X^2 - n\sigma_X^2(1 + \varepsilon_1^*)\right)/(N-n) + B_X\right)}{(A_X S_X^2 + B_X) + \left(A_X\left(NS_X^2 - n\sigma_X^2(1 + \varepsilon_1^*)\right)/(N-n) + B_X\right)}\right) \quad (35)$$

As $N \to \infty$, $S_X^2 \approx \sigma_X^2$, then equation (35) becomes (36)

$$\Upsilon_{2j} = \left(\sigma_Y^2 + \sigma_Y^2\varepsilon_0^* - b_{rg}\sigma_X^2\varepsilon_1^*\right) \exp\left(\frac{(A_X\sigma_X^2 + B_X) - (A_X\sigma_X^2 + B_X - A_X\sigma_X^2 n\varepsilon_1^*/(N-n))}{(A_X\sigma_X^2 + B_X) + (A_X\sigma_X^2 + B_X - A_X\sigma_X^2 n\varepsilon_1^*/(N-n))}\right) \quad (36)$$

By simplifying (36), equation (37) is obtained

$$\Upsilon_{2j} - \sigma_Y^2 = \sigma_Y^2\varepsilon_0^* + \left(\frac{n\lambda_k\sigma_Y^2}{2(N-n)} - b_{rg}\sigma_X^2\right)\varepsilon_1^* + \left(\frac{3}{8}\left(\frac{n\lambda_k}{N-n}\right)^2\sigma_Y^2 - \frac{b_{rg}\sigma_X^2 n\lambda_k}{2(N-n)}\right)\varepsilon_1^{*2} + \frac{n\lambda_k\sigma_Y^2\varepsilon_0^*\varepsilon_1^*}{2(N-n)} \quad (37)$$

By taking expectation of both sides of (37), the bias of $\Upsilon_{2j}, j = 1,2,\ldots,17$obtained as

$$Bias(\Upsilon_{2j}) = \left(\frac{3}{8}\left(\frac{n\lambda_k}{N-n}\right)^2\sigma_Y^2 - \frac{b_{rg}\sigma_X^2 n\lambda_k}{2(N-n)}\right)H_1 + \frac{n\lambda_k\sigma_Y^2}{2(N-n)}H_{01} \quad (38)$$

Similarly, by squaring (37) and take expectation of the results, the MSE of $\Upsilon_{2j},\ j = 1,2,\ldots,17$ is obtained as

$$MSE(\Upsilon_{2j}) = \sigma_Y^2 H_0 + \left(\frac{n\sigma_Y^2\lambda_k}{2(N-n)} - b_{rg}\sigma_X^2\right)^2 H_1 + 2\sigma_Y^2\left(\frac{n\sigma_Y^2\lambda_k}{2(N-n)} - b_{rg}\sigma_X^2\right)H_{01} \quad (39)$$

The estimators $T_{3j}, j = 1,2,3,\ldots,17$ can be expressed in terms of errors $\xi_i^*, i = 0,1$ as

$$\Upsilon_{3j} = \frac{\left((1 + \varepsilon_0^*)\sigma_Y^2 + b_{rg}(S_X^2 - (1 + \varepsilon_1^*)\sigma_X^2)\right)}{(A_X\,(NS_X^2 - n(1 + \varepsilon_1^*)\sigma_Y^2)/(N-n) + B_X)}(A_X S_X^2 + B_X) \times \exp\left(\frac{(A_X S_X^2 + B_X) - \left(A_X\left(NS_X^2 - n\sigma_X^2(1 + \varepsilon_1^*)\right)/(N-n) + B_X\right)}{(A_X S_X^2 + B_X) + \left(A_X\left(NS_X^2 - n\sigma_X^2(1 + \varepsilon_1^*)\right)/(N-n) + B_X\right)}\right)$$

As $N \to \infty$, $S_X^2 \approx \sigma_X^2$, then equation (40) becomes equation (41)

$$\Upsilon_{3j} = \frac{\left(\sigma_Y^2 + \sigma_Y^2\varepsilon_0^* - b_{rg}\sigma_X^2\varepsilon_1^*\right)}{1 - n\sigma_X^2\varepsilon_1^*/(A_X\sigma_X^2 + B_X)(N-n)} \times \exp\left(\frac{(A_X\sigma_X^2 + B_X) - (A_X\sigma_X^2 + B_X - A_X\sigma_X^2 n\varepsilon_1^*/(N-n))}{(A_X\sigma_X^2 + B_X) + (A_X\sigma_X^2 + B_X - A_X\sigma_X^2 n\varepsilon_1^*/(N-n))}\right)$$

By simplifying (41), equation (42) is obtained as

$$\begin{aligned}\Upsilon_{3j} &= \left(\sigma_Y^2 + \sigma_Y^2\varepsilon_0^* - b_{rg}\sigma_X^2\varepsilon_1^*\right)(1 - n\sigma_X^2\varepsilon_1^*/(N-n))^{-1}\exp\left(\frac{n\varepsilon_1^*\lambda_k}{2(N-n)} + \left(\frac{n\varepsilon_1^*\lambda_k}{2(N-n)}\right)^2\right) \quad (42)\end{aligned}$$

Simplify (42) to first order approximation, (43) is obtained

$$\Upsilon_{3j} - \sigma_Y^2 = \sigma_Y^2\varepsilon_0^* + \left(\frac{3n\sigma_Y^2\lambda_k}{2(N-n)} - b_{rg}\sigma_X^2\right)\varepsilon_1^* + \frac{n\lambda_k}{N-n}\left(\frac{15n\sigma_Y^2\lambda_k}{8(N-n)} - \frac{3}{2}b_{rg}\sigma_X^2\right)\varepsilon_1^{*2} + \frac{3n\sigma_Y^2\lambda_k}{2(N-n)}\varepsilon_0^*\varepsilon_1^* \quad (43)$$

By taking expectation of both sides of (43), the bias of $T_{3j}, j = 1,2,3,..,17$ is obtained as

$$Bias\left(\Upsilon_{3j}\right) = \frac{n\lambda_k}{N-n}\left(\frac{15n\sigma_Y^2\lambda_k}{8(N-n)} - \frac{3}{2}b_{rg}\sigma_X^2\right)H_1 + \frac{3n\sigma_Y^2\lambda_k}{2(N-n)}H_{01} \quad (44)$$

Similarly, by squaring (43) and take expectation of the results, the MSE of $T_{3j}, j = 1,2,3,..,17$ is obtained as

$$MSE\left(\Upsilon_{3j}\right) = \sigma_Y^4 H_0 + \left(\frac{3n\sigma_Y^2\lambda_k}{2(N-n)} - b_{rg}\sigma_X^2\right)^2 H_1 + 2\sigma_Y^2\left(\frac{3n\sigma_Y^2\lambda_k}{2(N-n)} - b_{rg}\sigma_X^2\right)^2 H_{01} \quad (45)$$

3.4 Theoretical Efficiency Comparison

In this section, efficiency conditions of the new variance estimators over some existing variance estimators considered in the study were established.

3.4.1 Efficiency Conditions of $\Upsilon_{1j}, j = 1,2,3,..,17$ ***Over Some Existing Estimators***

(i) $MSE\left(\Upsilon_{1j}\right) - MSE(\hat{t}_e^*) < 0,\ j = 1,2,3,..,17 \therefore \frac{H_1}{H_{01}} < -\sigma_Y^2/\left(\frac{n\lambda_k\sigma_Y^2}{N-n} - b_{rg}\sigma_X^2\right)$ (46)

(ii) $MSE\left(\Upsilon_{1j}\right) - MSE(\hat{t}_1^*) < 0,\ j = 1,2,3,..,17 \therefore \frac{H_1}{H_{01}} < -2\sigma_Y^2/\left(\left(\frac{n\lambda_k}{N-n} - 1\right)\sigma_Y^2 - b_{rg}\sigma_X^2\right)$ (47)

(iii) $MSE\left(\Upsilon_{1j}\right) - MSE(\hat{t}_2^*) < 0,\ j = 1,2,3,..,17 \therefore \frac{H_1}{H_{01}} < -2\sigma_Y^2/\left(\left(\frac{n\lambda_k}{N-n} - \frac{1}{2}\right)\sigma_Y^2 - b_{rg}\sigma_X^2\right)$ (48)

(iv) $MSE\left(\Upsilon_{1j}\right) - MSE(\hat{t}_3^*) < 0,\ j = 1,2,3,..,17 \therefore \frac{H_1}{H_{01}} < -\sigma_Y^2/\left(\frac{n\lambda_k\sigma_Y^2}{N-n} - b_{rg}\sigma_X^2\right)$ (49)

(v) $MSE\left(\Upsilon_{1j}\right) - MSE(\hat{t}_4^*) < 0,\ j = 1,2,3,..,17$

$$\left(\frac{n\lambda_k\sigma_Y^2}{N-n} - b_{rg}\sigma_X^2\right)^2 H_1 + 2\sigma_Y^2\left(\frac{n\lambda_k\sigma_Y^2}{N-n} - b_{rg}\sigma_X^2\right)H_{01} < \sigma_Y^4(0.25H_2 - H_{02}) \quad (50)$$

(vi) $MSE\left(\Upsilon_{1j}\right) - MSE(\hat{t}_{i1}^*) < 0,\ j = 1,2,3,..,17$

$$\left(\frac{n\lambda_k\sigma_Y^2}{N-n} - b_{rg}\sigma_X^2\right)^2 H_1 + 2\sigma_Y^2\left(\frac{n\lambda_k\sigma_Y^2}{N-n} - b_{rg}\sigma_X^2\right)H_{01} < \sigma_Y^4(\theta_i^2\Psi_i^2 H_i - 2\theta_i\Psi_i H_{0i}) \quad (51)$$

(vii) $MSE\left(\Upsilon_{1j}\right) - MSE(\hat{t}_{i2}^*) < 0,\ j = 1,2,3,..,17$

$$\left(\frac{n\lambda_k\sigma_Y^2}{N-n} - b_{rg}\sigma_X^2\right)^2 H_1 + 2\sigma_Y^2\left(\frac{n\lambda_k\sigma_Y^2}{N-n} - b_{rg}\sigma_X^2\right)H_{01} < \sigma_Y^4(\vartheta_i^2\Psi_i^2 H_i - 2\vartheta_i\Psi_i H_{0i}) \quad (52)$$

(viii) $MSE\left(\Upsilon_{1j}\right) - MSE(\hat{t}_{i3}^*) < 0,\ j = 1,2,3,..,17$

$$\left(\frac{n\lambda_k\sigma_Y^2}{N-n} - b_{rg}\sigma_X^2\right)^2 H_1 + 2\sigma_Y^2\left(\frac{n\lambda_k\sigma_Y^2}{N-n} - b_{rg}\sigma_X^2\right)H_{01} < \sigma_Y^4(2^{-2}\phi_i^2\Psi_i^2 H_i - \phi_i\Psi_i H_{0i}) \quad (53)$$

If the conditions obtained in equation (46) to (53) respectively are satisfied, then the proposed class of variance estimators $\Upsilon_{1j}, j = 1,2,3,..,17$ are more efficient and can produce better estimate of population variance in the presence of non-response and measurement errors than existing estimators $\hat{t}_{0e}^*, \hat{t}_i^*(i = 1,2,3,4), \hat{t}_{i1}^*, \hat{t}_{i2}^*, \hat{t}_{i3}^*$ in (1), (4), (7), (10), (13), (16), (17) and (18) respectively proposed by Isaki (983), Singh et al. (2011), Yadav and Kadilar (2013), Asghar et al. (2015), Audu et al. (2022) respectively.

3.4.2 Efficiency Conditions of $\Upsilon_{2j}, j = 1,2,3,..,17$ ***Over Some Existing Estimators***

(i) $MSE\left(\Upsilon_{2j}\right) - MSE(\hat{t}_e^*) < 0,\ j = 1,2,3,..,17 \therefore \frac{H_1}{H_{01}} < -2\sigma_Y^2/\left(\frac{n\lambda_k\sigma_Y^2}{2(N-n)} - b_{rg}\sigma_X^2\right)$ (54)

(ii) $MSE\left(\Upsilon_{2j}\right) - MSE(\hat{t}_1^*) < 0,\ j = 1,2,3,..,17 \therefore \frac{H_1}{H_{01}} < -2\sigma_Y^2/\left(\left(\frac{n\lambda_k}{2(N-n)} - 1\right)\sigma_Y^2 - b_{rg}\sigma_X^2\right)$ (55)

(iii) $MSE\left(\Upsilon_{2j}\right) - MSE(\hat{t}_2^*) < 0,\ j = 1,2,3,..,17 \therefore \frac{H_1}{H_{01}} < -2\sigma_Y^2/\left(\left(\frac{n\lambda_k}{2(N-n)} - \frac{1}{2}\right)\sigma_Y^2 - b_{rg}\sigma_X^2\right)$ (56)

(iv) $MSE(\Upsilon_{2j}) - MSE(\hat{t}_3^*) < 0,\ j = 1,2,3,..,17 \therefore \frac{H_1}{H_{01}} < -\sigma_Y^2/\left(\frac{n\lambda_k\sigma_Y^2}{2(N-n)} - b_{rg}\sigma_X^2\right)$ (57)

(v) $MSE(\Upsilon_{2j}) - MSE(\hat{t}_4^*) < 0,\ j = 1,2,3,..,17$

$$\left(\frac{n\lambda_k\sigma_Y^2}{2(N-n)} - b_{rg}\sigma_X^2\right)^2 H_1 + 2\sigma_Y^2\left(\frac{n\lambda_k\sigma_Y^2}{2(N-n)} - b_{rg}\sigma_X^2\right)H_{01} < \sigma_Y^4(0.25H_2 - H_{02}) \quad (58)$$

(vi) $MSE(\Upsilon_{2j}) - MSE(\hat{t}_{i1}^*) < 0,\ j = 1,2,3,..,17$

$$\left(\frac{n\lambda_k\sigma_Y^2}{2(N-n)} - b_{rg}\sigma_X^2\right)^2 H_1 + 2\sigma_Y^2\left(\frac{n\lambda_k\sigma_Y^2}{2(N-n)} - b_{rg}\sigma_X^2\right)H_{01} < \sigma_Y^4(\theta_i^2\Psi_i^2 H_i - 2\theta_i\Psi_i H_{0i}) \quad (59)$$

(vii) $MSE(\Upsilon_{2j}) - MSE(\hat{t}_{i2}^*) < 0,\ j = 1,2,3,..,17$

$$\left(\frac{n\lambda_k\sigma_Y^2}{2(N-n)} - b_{rg}\sigma_X^2\right)^2 H_1 + 2\sigma_Y^2\left(\frac{n\lambda_k\sigma_Y^2}{2(N-n)} - b_{rg}\sigma_X^2\right)H_{01} < \sigma_Y^4(\vartheta_i^2\Psi_i^2 H_i - 2\vartheta_i\Psi_i H_{0i}) \quad (60)$$

(viii) $MSE(\Upsilon_{2j}) - MSE(\hat{t}_{i3}^*) < 0,\ j = 1,2,3,..,17$

$$\left(\frac{n\lambda_k\sigma_Y^2}{2(N-n)} - b_{rg}\sigma_X^2\right)^2 H_1 + 2\sigma_Y^2\left(\frac{n\lambda_k\sigma_Y^2}{2(N-n)} - b_{rg}\sigma_X^2\right)H_{01} < \sigma_Y^4(2^{-2}\phi_i^2\Psi_i^2 H_i - \phi_i\Psi_i H_{0i}) \quad (61)$$

If the conditions obtained in equation (54) to (61) respectively are satisfied, then the proposed class of variance estimators $\Upsilon_{2j}, j = 1,2,3,..,17$ are more efficient and can produce better estimate of population variance in the presence of non-response and measurement errors than existing estimators $\hat{t}_{0e}^*, \hat{t}_i^*(i = 1,2,3,4), \hat{t}_{i1}^*, \hat{t}_{i2}^*, \hat{t}_{i3}^*$ in (1), (4), (7), (10), (13), (16), (17) and (18) respectively proposed by Isaki (983), Singh et al. (2011), Yadav and Kadilar (2013), Asghar et al. (2015), Audu et al. (2022) respectively.

3.4.3 Efficiency Conditions of $\Upsilon_{3j}, j = 1,2,3,..,17$ Over Some Existing Estimators

(i) (i) $MSE(\Upsilon_{3j}) - MSE(\hat{t}_e^*) < 0,\ j = 1,2,3,..,17 \therefore \frac{H_1}{H_{01}} < -2\sigma_Y^2/\left(\frac{3n\lambda_k\sigma_Y^2}{2(N-n)} - b_{rg}\sigma_X^2\right)$ (62)

(ii) $MSE(\Upsilon_{3j}) - MSE(\hat{t}_1^*) < 0,\ j = 1,2,3,..,17 \therefore \frac{H_1}{H_{01}} < -2\sigma_Y^2/\left(\left(\frac{3n\lambda_k}{2(N-n)} - 1\right)\sigma_Y^2 - b_{rg}\sigma_X^2\right)$ (63)

(iii) $MSE(\Upsilon_{3j}) - MSE(\hat{t}_2^*) < 0,\ j = 1,2,3,..,17 \therefore \frac{H_1}{H_{01}} < -2\sigma_Y^2/\left(\left(\frac{n\lambda_k}{2(N-n)} - \frac{1}{2}\right)\sigma_Y^2 - b_{rg}\sigma_X^2\right)$ (64)

(iv) $MSE(\Upsilon_{3j}) - MSE(\hat{t}_3^*) < 0,\ j = 1,2,3,..,17 \therefore \frac{H_1}{H_{01}} < -\sigma_Y^2/\left(\frac{3n\lambda_k\sigma_Y^2}{2(N-n)} - b_{rg}\sigma_X^2\right)$ (65)

(v) $MSE(\Upsilon_{3j}) - MSE(\hat{t}_4^*) < 0,\ j = 1,2,3,..,17$

$$\left(\frac{3n\lambda_k\sigma_Y^2}{2(N-n)} - b_{rg}\sigma_X^2\right)^2 H_1 + 2\sigma_Y^2\left(\frac{3n\lambda_k\sigma_Y^2}{2(N-n)} - b_{rg}\sigma_X^2\right)H_{01} < \sigma_Y^4(0.25H_2 - H_{02}) \quad (66)$$

(vi) $MSE(\Upsilon_{3j}) - MSE(\hat{t}_{i1}^*) < 0,\ j = 1,2,3,..,17$

$$\left(\frac{3n\lambda_k\sigma_Y^2}{2(N-n)} - b_{rg}\sigma_X^2\right)^2 H_1 + 2\sigma_Y^2\left(\frac{3n\lambda_k\sigma_Y^2}{2(N-n)} - b_{rg}\sigma_X^2\right)H_{01} < \sigma_Y^4(\theta_i^2\Psi_i^2 H_i - 2\theta_i\Psi_i H_{0i}) \quad (67)$$

(vii) $MSE(\Upsilon_{3j}) - MSE(\hat{t}_{i2}^*) < 0,\ j = 1,2,3,..,17$

$$\left(\frac{3n\lambda_k\sigma_Y^2}{2(N-n)} - b_{rg}\sigma_X^2\right)^2 H_1 + 2\sigma_Y^2\left(\frac{3n\lambda_k\sigma_Y^2}{2(N-n)} - b_{rg}\sigma_X^2\right)H_{01} < \sigma_Y^4(\vartheta_i^2\Psi_i^2 H_i - 2\vartheta_i\Psi_i H_{0i}) \quad (68)$$

(viii) $MSE(\Upsilon_{3j}) - MSE(\hat{t}_{i3}^*) < 0,\ j = 1,2,3,..,17$

$$\left(\frac{n\lambda_k\sigma_Y^2}{2(N-n)} - b_{rg}\sigma_X^2\right)^2 H_1 + 2\sigma_Y^2\left(\frac{n\lambda_k\sigma_Y^2}{2(N-n)} - b_{rg}\sigma_X^2\right)H_{01} < \sigma_Y^4(2^{-2}\phi_i^2\Psi_i^2 H_i - \phi_i\Psi_i H_{0i}) \quad (69)$$

If the conditions obtained in equation (62) to (69) respectively are satisfied, then the proposed class of variance estimators $\Upsilon_{3j}, j = 1,2,3,..,17$ are more efficient and can produce better estimate of population variance in the presence of non-response and measurement errors than existing estimators $\hat{t}_{0e}^*, \hat{t}_i^*(i = 1,2,3,4), \hat{t}_{i1}^*, \hat{t}_{i2}^*, \hat{t}_{i3}^*$ in (1), (4), (7), (10), (13), (16), (17) and (18) respectively proposed by Isaki (983), Singh et al. (2011), Yadav and Kadilar (2013), Asghar et al. (2015), Audu et al. (2022) respectively.

4. Empirical Study

To measure the efficiency (accuracy and precision) of the proposed estimators with respect to proposed traditional estimator. The bias and mean square error (MSE) of the estimators were computed using (70) and (71) respectively.

$Bias\ (\theta) = E(\theta) - \bar{Y}$ (70)

$MSE(\theta) = E(\theta - \bar{Y})^2$ (71)

where θ is any estimator considered in the study.

Simulation studies were conducted to evaluate the superiority of the proposed estimators compared to other estimators considered in the present study. For this purpose, 5 different populations of size $N = 1000$ were generated. The data for study and auxiliary variables were generated using the distributions defined in Table 4. A sample of size $n = 100$ was selected 1000 times using sampling without replacement method form each population generated for computations of biases, MSEs and PREs of the estimators under consideration.

Table 4: Distributions of Populations used for Empirical Study

Popn	Auxiliary variable	Study variable
I	$\boldsymbol{Z_2 = abs\left(\frac{1}{100\sqrt{2\pi}} exp\left(\frac{1}{2}\left(\frac{X-10}{100}\right)^2\right)\right), -\infty < X < \infty}$ $Z_1 = \frac{1}{B(\alpha,\beta)} X^{\alpha-1}(1-X)^{\beta-1}, \alpha = 1, \beta = 1$ $B(\alpha,\beta) = \int_0^1 X^{\alpha-1}(1-X)^{\beta-1} dX, \quad 0 \le X < 1$ $\boldsymbol{X = \rho_{z_1z_2} Z_1 + \sqrt{1-\rho_{z_1z_2}} Z_2}$	$Y = X_i + \varepsilon_i$ ε: $abs(N(0,1))$
II	$\boldsymbol{Z_1 = \frac{1}{b-a}, a \le Z_1 \le b, a = 1, b = 100}$ $X = \rho_{z_1z_2} Z_1 + \sqrt{1-\rho_{z_1z_2}} Z_2$	
III	$\boldsymbol{Z_1 = \frac{1}{\Gamma(\alpha)} \lambda^\alpha X^{\alpha-1} e^{-\lambda X}, \alpha = 3, \lambda = 1, X \ge 0}$ $X = \rho_{z_1z_2} Z_1 + \sqrt{1-\rho_{z_1z_2}} Z_2$	
IV	$\boldsymbol{Z_1 = \lambda e^{-\lambda x}, \lambda = 1.5, X \ge 0}$ $\boldsymbol{X = \rho_{z_1z_2} Z_1 + \sqrt{1-\rho_{z_1z_2}} Z_2}$	
V	$\boldsymbol{Z_1 = \frac{1}{x!} \lambda^x e^{-\lambda}, \lambda = 0.1, X \ge 0}$ $\boldsymbol{X = \rho_{z_1z_2} Z_1 + \sqrt{1-\rho_{z_1z_2}} Z_2}$	

The study variable $\boldsymbol{Y}$ was inflated by measurement error which follows normal distribution with mean zero and variance 4 while the auxiliary variable $\boldsymbol{X}$ was inflated by measurement error which follows normal distribution with mean zero and variance 25.

Table 5: Biases, MSEs and PREs of the Proposed and Existing Variance Estimators using Population I

Estimators	Bias	MSE	PRE
Sample variance $s^{*2}_{y(e)}$	-12.87299	501558.9	100
Isaki (1983) $\hat{t}^*_1$	0.3972484	328214.5	152.8144
Singh et al (2011) $\hat{t}^*_2$	-20.35941	128807.9	389.3852
Yadav and Kadilar (2013) $\hat{t}^*_3$	-28.95013	60898.04	823.6044
Asghar et al. (2014) $\hat{t}^*_4$	2.189216	645578.8	77.69135
Audu et al. (2022)			
$\hat{t}^*_{11(1)}$	699.6159	1052203	47.66752
$\hat{t}^*_{11(2)}$	373.2478	662432	75.71477
$\hat{t}^*_{21(1)}$	-19.52166	21329.1	2351.524
$\hat{t}^*_{21(2)}$	-20.49848	23767.82	2110.243

$\hat{t}^*_{12(1)}$	-24.79059	409300	122.5406
$\hat{t}^*_{12(2)}$	-23.83408	429611.2	116.7472
$\hat{t}^*_{22(1)}$	-19.48857	21846.48	2295.834
$\hat{t}^*_{22(2)}$	-20.46132	24295.1	2064.445
$\hat{t}^*_{13(1)}$	-24.78484	409281.9	122.5461
$\hat{t}^*_{13(2)}$	-23.82389	429643.6	116.7384
$\hat{t}^*_{23(1)}$	-20.13417	23511.66	2133.235
$\hat{t}^*_{23(2)}$	-21.05096	25977.3	1930.758
Proposed Estimators			
Υ_{11}	-0.3713696	9160.292	5475.359
Υ_{12}	-0.3901421	8981.528	5584.338
Υ_{13}	-0.3716061	9158.058	5476.695
Υ_{21}	-0.5094332	9160.292	5475.359
Υ_{22}	-0.5138308	4549.368	11024.8
Υ_{23}	-0.5094891	4592.334	10921.66
Υ_{31}	0.1088907	16891.97	2969.215
Υ_{32}	0.06577789	16485.08	3042.503
Υ_{33}	0.1083491	16886.88	2970.11

Table 5 shows that, under Population I, every proposed estimator was more efficient than every existing estimator. The sample variance estimator, the benchmark, had an MSE of 501,558.9. Among the conventional competitors, Isaki (1983), Singh et al. (2011), and Yadav and Kadilar (2013) lowered the MSE to 328,214.5, 128,807.9, and 60,898.04 (PRE = 152.81, 389.39, and 823.60), the last being the best of the four. The estimator of Asghar et al. (2014) and the Audu et al. (2022) members $\hat{t}^*_{11(1)}$ and $\hat{t}^*_{11(2)}$ had PRE values below 100 (77.69, 47.67, and 75.71), that is, they were less efficient than the sample variance, and the latter two were also heavily biased (699.62 and 373.25). The best Audu et al. member, $\hat{t}^*_{21(1)}$, reached an MSE of 21,329.1 (PRE = 2,351.52). The proposed estimators had MSEs between 4,549.37 and 16,891.97 and PREs between 2,969.22 and 11,024.80, so even the weakest (Υ_{31}) exceeded the best existing estimator. The most efficient was Υ_{22} (MSE = 4,549.37; PRE = 11,024.80), which reduced the MSE by about 99% relative to the sample variance and to roughly one-fifth of that of $\hat{t}^*_{(21(1))}$. Within each class, the member with $B_X = S_X(\Upsilon_{12}, \Upsilon_{22}, \Upsilon_{32})$ was the most efficient. The absolute biases of the proposed estimators did not exceed 0.52, against 12.87 for the sample variance, so the squared bias is negligible relative to the MSE and the gain in precision was not obtained at the expense of accuracy. Υ_{11} and Υ_{21} produced identical MSE and PRE values.

Table 6: Biases, MSEs and PREs of the Proposed and Existing Variance Estimators using Population II

Estimators	Bias	MSE	PRE
Sample variance $s^{*2}_{y(e)}$	2.77563	371231.6	100
Isaki (1983) $\hat{t}^*_1$	2.579833	289281.9	128.3286
Singh et al (2011) $\hat{t}^*_2$	-8.670112	94569.23	392.5501
Yadav and Kadilar (2013) $\hat{t}^*_3$	-17.41334	44198.41	839.9207
Asghar et al. (2014) $\hat{t}^*_4$	8.804765	491683.8	75.5021
Audu et al. (2022)			
$\hat{t}^*_{11(1)}$	-601.8709	611840.6	60.67455
$\hat{t}^*_{11(2)}$	-337.6506	362708.1	102.35
$\hat{t}^*_{21(1)}$	15.77179	10056.89	3691.315
$\hat{t}^*_{21(2)}$	13.79608	8262.836	4492.786
$\hat{t}^*_{12(1)}$	15.09768	305207.8	121.6324
$\hat{t}^*_{12(2)}$	1.26495	299269.9	124.0457
$\hat{t}^*_{22(1)}$	15.69666	10288.6	3608.183
$\hat{t}^*_{22(2)}$	13.7204	8453.209	4391.606
$\hat{t}^*_{13(1)}$	14.98401	304965	121.7292
$\hat{t}^*_{13(2)}$	1.266161	299322.1	124.0241
$\hat{t}^*_{23(1)}$	14.99155	8721.058	4256.726
$\hat{t}^*_{23(2)}$	13.0939	7181.252	5169.455

Proposed Estimators			
Υ_{11}	3.459424	7553.332	4914.805
Υ_{12}	3.423917	7410.04	5009.846
Υ_{13}	3.45896	7551.459	4916.024
Υ_{21}	2.713226	7553.332	4914.805
Υ_{22}	2.699753	4001.986	9276.184
Υ_{23}	2.71305	4037.265	9195.125
Υ_{31}	4.485267	13431.29	2763.931
Υ_{32}	4.419178	13107.58	2832.19
Υ_{33}	4.484404	13427.06	2764.802

Table 6 reports the results for Population II, where the sample variance estimator had an MSE of 371,231.6. Yadav and Kadilar (2013) was again the best conventional estimator (MSE = 44,198.41; PRE = 839.92), whereas Asghar et al. (2014) and t11(1)* were less efficient than the benchmark (PRE = 75.50 and 60.67). The Audu et al. (2022) class improved markedly, with $t^*_{23(2)}$ reaching an MSE of 7,181.25 and a PRE of 5,169.46. The outcome was less uniform among the proposed estimators than in Population I. The exponential-type estimators Y22 and Υ_{23} had the smallest MSEs of all estimators (4,001.99 and 4,037.27) and the highest PREs (9,276.18 and 9,195.13), about 1.8 times more precise than $t^*_{23(2)}$ and about 92 times more precise than the sample variance. The ratio-type estimators $\Upsilon_{11}, \Upsilon_{12}, \Upsilon_{13}$, and Υ_{21} (PRE = 4,914.81 to 5,009.85) surpassed all existing estimators except $t^*_{23(2)}$, while the product-type estimators $\Upsilon_{31}, \Upsilon_{32}$, and Υ_{33} (PRE = 2,763.93 to 2,832.19) were surpassed by six Audu et al. members ($t^*_{21(1)}, t^*_{21(2)}, t^*_{22(1)}, t^*_{22(2)}, t^*_{23(1)}$, and $t^*_{23(2)}$). The superiority of the proposed estimators in this population is therefore tied mainly to the exponential structure of the Υ_{2j} class and not to every proposed form. Their biases (2.70 to 4.49) were comparable with that of the sample variance (2.78) and, although larger than those of $t^*_{12(2)}$ and $t^*_{13(2)}$ (about 1.27), accounted for less than 0.2% of the MSE

Table 7: Biases, MSEs and PREs of the Proposed and Existing Variance Estimators using Population III

Estimators	Bias	MSE	PRE
Sample variance $s^{*2}_{y(e)}$	41.7483	562845.4	100
Isaki (1983) $\hat{t}^*_1$	-2.490038	70304.71	800.5799
Singh et al (2011) $\hat{t}^*_2$	4.175094	143325.2	392.705
Yadav and Kadilar (2013) $\hat{t}^*_3$	-15.03812	66364.81	848.1082
Asghar et al. (2014) $\hat{t}^*_4$	55.07883	745530.5	75.49595
Audu et al. (2022)			
$\hat{t}^*_{11(1)}$	166.9323	482603.5	116.6269
$\hat{t}^*_{11(2)}$	100.8607	481254.9	116.9537
$\hat{t}^*_{21(1)}$	-4.650289	12731.08	4421.034
$\hat{t}^*_{21(2)}$	-5.685194	19373.23	2905.273
$\hat{t}^*_{12(1)}$	39.55445	432459.1	130.15
$\hat{t}^*_{12(2)}$	32.59179	455575.9	123.5459
$\hat{t}^*_{22(1)}$	-4.60941	12922.39	4355.582
$\hat{t}^*_{22(2)}$	-5.644807	19647.71	2864.687
$\hat{t}^*_{13(1)}$	39.39246	432336.7	130.1868
$\hat{t}^*_{13(2)}$	32.58771	455645.9	123.5269
$\hat{t}^*_{23(1)}$	-5.103851	20333.98	2768.003
$\hat{t}^*_{23(2)}$	-6.061903	29056.23	1937.09
Proposed Estimators			
Υ_{11}	4.01244	8324.227	6761.533
Υ_{12}	3.895049	8109.38	6940.671
Υ_{13}	4.0109	8321.397	6763.833
Υ_{21}	0.5821764	8324.227	6761.533
Υ_{22}	0.5291727	2701.521	20834.39
Υ_{23}	0.5814816	2756.333	20420.08
Υ_{31}	7.829697	17461.21	3223.404
Υ_{32}	7.636563	16980.83	3314.593
Υ_{33}	7.827163	17454.88	3224.572

Table 7 shows that the sample variance estimator had an MSE of 562,845.4 under Population III. The conventional estimators of Isaki (1983) and Yadav and Kadilar (2013) were both effective here (MSE = 70,304.71 and 66,364.81; PRE = 800.58 and 848.11), whereas Singh et al. (2011) reached PRE = 392.71 and Asghar et al. (2014) fell below the benchmark (PRE = 75.50). The best Audu et al. (2022) member, $t^*_{21(1)}$, had an MSE of 12,731.08 (PRE = 4,421.03); it was the only existing estimator with a PRE above 4,400. The ratio-type estimators Y_{11}, Y_{12}, Y_{13}, and Y_{21} (PRE = 6,761.53 to 6,940.67) and the exponential-type estimators Y_{22} and Y_{23} (PRE = 20,834.39 and 20,420.08) exceeded every existing estimator. Y_{22} had the smallest MSE (2,701.52), about 4.7 times lower than that of $t^*_{21(1)}$ and 99.5% lower than that of the sample variance, the largest gain in the study together with Population V. The product-type estimators Y_{31}, Y_{32}, and Y_{33} (PRE = 3,223.40 to 3,314.59) were less efficient than $t^*_{21(1)}$ and $t^*_{22(1)}$ but still outperformed 14 of the 16 existing estimators. The biases of Y_{21}, Y_{22}, and Y_{23} (0.53 to 0.58) were smaller than that of any existing estimator, while those of the Y_{1j} and Y_{3j} members (about 3.9 to 4.0 and 7.6 to 7.8) were moderate and contributed no more than 0.35% of the MSE.

Table 8: Biases, MSEs and PREs of the Proposed and Existing Variance Estimators using Population IV

Estimators	Bias	MSE	PRE
Sample variance $s^{*2}_{y(e)}$	254.8369	707755.5	100
Isaki (1983) $\hat{t}^*_1$	228.6473	532934.0	132.8036
Singh et al (2011) $\hat{t}^*_2$	225.4323	209114.9	338.4529
Yadav and Kadilar (2013) $\hat{t}^*_3$	208.498	112917.9	626.7875
Asghar et al. (2014) $\hat{t}^*_4$	277.2445	923744.9	76.61807
Audu et al. (2022)			
$\hat{t}^*_{11(1)}$	-1510.709	2600255	27.2187
$\hat{t}^*_{11(2)}$	-720.0603	814279	86.91807
$\hat{t}^*_{21(1)}$	270.4577	117115	604.3253
$\hat{t}^*_{21(2)}$	267.1777	109477.6	646.4842
$\hat{t}^*_{12(1)}$	237.8398	525202.5	134.7586
$\hat{t}^*_{12(2)}$	233.1073	539809.5	131.1121
$\hat{t}^*_{22(1)}$	269.1598	118933.2	595.0864
$\hat{t}^*_{22(2)}$	265.9155	111042.8	637.3717
$\hat{t}^*_{13(1)}$	237.7684	524708.5	134.8855
$\hat{t}^*_{13(2)}$	233.1299	539951.3	131.0777
$\hat{t}^*_{23(1)}$	270.5063	111239.1	636.2469
$\hat{t}^*_{23(2)}$	267.315	104351.4	678.2425
Proposed Estimators			
Y_{11}	233.1406	63217.27	1119.56
Y_{12}	233.0502	62944.42	1124.413
Y_{13}	233.1395	63213.89	1119.62
Y_{21}	230.6224	63217.27	1119.56
Y_{22}	230.5834	56011.62	1263.587
Y_{23}	230.6219	56084.36	1261.948
Y_{31}	236.0891	74773.98	946.5265
Y_{32}	235.9349	74171.84	954.2106
Y_{33}	236.0872	74766.52	946.621

Table 8 shows a different pattern for Population IV. The sample variance estimator had an MSE of 707,755.5 and a bias of 254.84, and nearly all estimators, proposed and existing, were substantially biased (absolute bias of about 208 to 277, apart from $t^*_{11(1)}$ and $t^*_{11(2)}$ at −1,510.71 and −720.06). Yadav and Kadilar (2013) was the best conventional estimator (PRE = 626.79), and the best Audu et al. (2022) member, $t^*_{23(2)}$, had an MSE of 104,351.4 (PRE = 678.24); Asghar et al. (2014), $t^*_{11(1)}$, and $t^*_{11(2)}$ had PRE values below 100 (76.62, 27.22, and 86.92). All nine proposed estimators had smaller MSEs than every existing estimator (56,011.62 to 74,773.98; PRE = 946.53 to 1,263.59), with Y_{22} the most efficient, about 1.9 times more precise than $t^*_{23(2)}$ and 92% lower in MSE than the sample variance. The efficiency gains were nevertheless much smaller than in the other populations, and the proposed biases (230.58 to 236.09) were close to those of existing estimators. The squared bias accounted for roughly 75% to 95% of the proposed estimators' MSE (for example, 94.9% for Y_{22}), against about 38% for Yadav and Kadilar (2013). In this population the precision of the proposed

estimators is therefore limited mainly by bias rather than by variance, and further gains would require bias reduction.

Table 9: Biases, MSEs and PREs of the Proposed and Existing Variance Estimators using Population V

Estimators	Bias	MSE	PRE
Sample variance $s^{*2}_{y(e)}$	41.89738	563288.3	100
Isaki (1983) $\hat{t}^*_1$	-2.482284	70188.3	802.5387
Singh et al (2011) $\hat{t}^*_2$	4.234353	143436.5	392.7091
Yadav and Kadilar (2013) $\hat{t}^*_3$	-15.0169	66415.3	848.1304
Asghar et al. (2014) $\hat{t}^*_4$	55.11653	741796.1	75.93573
Audu et al. (2022)			
$\hat{t}^*_{11(1)}$	126.8444	463824.2	121.4444
$\hat{t}^*_{11(2)}$	80.0161	472948.1	119.1015
$\hat{t}^*_{21(1)}$	-3.976554	9970.581	5649.503
$\hat{t}^*_{21(2)}$	-5.0538	15205.3	3704.553
$\hat{t}^*_{12(1)}$	39.28937	432918.1	130.1143
$\hat{t}^*_{12(2)}$	32.67623	456178.2	123.4799
$\hat{t}^*_{22(1)}$	-3.948287	10064.65	5596.702
$\hat{t}^*_{22(2)}$	-5.025874	15358.96	3667.491
$\hat{t}^*_{13(1)}$	39.14044	432810.9	130.1465
$\hat{t}^*_{13(2)}$	32.67271	456247.2	123.4612
$\hat{t}^*_{23(1)}$	-4.453166	16162.07	3485.247
$\hat{t}^*_{23(2)}$	-5.451531	23599.37	2386.878
Proposed Estimators			
Υ_{11}	4.038537	8333.001	6759.73
Υ_{12}	3.920818	8117.857	6938.879
Υ_{13}	4.037021	8330.219	6761.987
Υ_{21}	0.5990315	8333.001	6759.73
Υ_{22}	0.5458734	2703.839	20832.91
Υ_{23}	0.5983476	2758.751	20418.23
Υ_{31}	7.865562	17479.49	3222.567
Υ_{32}	7.671907	16998.47	3313.758
Υ_{33}	7.863068	17473.27	3223.714

Table 9 reports results for Population V that closely mirror those of Table 7 for Population III, so the ranking of the estimators was the same in both. The sample variance estimator had an MSE of 563,288.3. Among the conventional estimators, Yadav and Kadilar (2013) was the most efficient (MSE = 66,415.3; PRE = 848.13), followed by Isaki (1983) (PRE = 802.54) and Singh et al. (2011) (PRE = 392.71), while Asghar et al. (2014) had a PRE of 75.94. The best Audu et al. (2022) member, $t_{21(1)}$ *, had an MSE of 9,970.58 (PRE = 5,649.50), followed by $t^*_{22(1)}$ (PRE = 5,596.70). The ratio-type estimators $\Upsilon_{11}, \Upsilon_{12}, \Upsilon_{13}$, and Υ_{21} (PRE = 6,759.73 to 6,938.88) and the exponential-type estimators Υ_{22} and Υ_{23} (PRE = 20,832.91 and 20,418.23) outperformed all existing estimators. Υ_{22} had the smallest MSE (2,703.84), about 3.7 times lower than that of $t^*_{21(1)}$ and 99.5% lower than that of the sample variance. The product-type estimators $\Upsilon_{31}, \Upsilon_{32}$, and Υ_{33} (PRE = 3,222.57 to 3,313.76) were less efficient than five Audu et al. members ($t^*_{21(1)}, t^*_{22(1)}, t^*_{21(2)}, t^*_{22(2)}$, and $t^*_{23(1)}$) but far more efficient than all conventional estimators (PRE of 848.13 or less). The biases of $\Upsilon_{21}, \Upsilon_{22}$, and Υ_{23} (0.55 to 0.60) were smaller than those of all existing estimators, whereas those of the Υ_{1j} and Υ_{3j} members (about 4.0 and 7.9) were moderate and contributed no more than 0.35% of the MSE.

5.0 Discussion of Results

Across all five populations, the sample variance estimator had the largest MSE (371,231.6 to 707,755.5) and served as the baseline with PRE=100. The proposed estimators reduced the MSE to between 2,701.5 and 74,774.0. The best member, Υ_{22}, attained PRE values of 11,024.8 (Population I), 9,276.2 (II), 20,834.4 (III), 1,263.6 (IV) and 20,832.9 (V). Its MSE was lower than that of the best conventional estimator (Yadav and Kadilar, 2013) by factors of about 13.4, 11.0, 24.6, 2.0 and 24.6 in Populations I to V respectively, and lower than that of the best Audu et al. (2022) member by factors of about 4.7, 1.8, 4.7, 1.9 and 3.7.

The three proposed classes were ranked consistently in every population. The exponential-type class Υ_{2j} was the most efficient, Υ_{1j} was intermediate, and the

product-exponential class Y_{3j} was the weakest, although it still performed far better than the conventional estimators (PRE of at least 946). This ordering is consistent with the MSE expressions in (34), (39) and (45), in which the classes differ in the multiplier applied to $n\lambda k/(N-n)$. Within each class, the choice $B_X = S_X$ (Y_{12}, Y_{22}, Y_{32}) gave the lowest MSE, $B_X = C_X$ was close behind, and $B_X = 0$ was slightly less efficient. The choice of class therefore matters more than the choice of (A_X, B_X).

Among the existing estimators, Yadav and Kadilar (2013) was the most efficient conventional estimator (PRE of about 627 to 848). The estimator of Asghar et al. (2014) had a PRE below 100 in every population, meaning it was less efficient than the sample variance; a plausible reason is that it uses information on the auxiliary mean rather than the auxiliary variance. The Audu et al. (2022) estimators were erratic: some members, such as $t_{21(1)}$, gave large gains, while others, such as $t_{11(1)}$, were unstable (bias of −1,510.7 and PRE of 27.2 in Population IV).

Regarding bias, the proposed estimators were small in absolute terms in Populations I to III (absolute bias below 8, and below 1 for Y_{2j} in Population I and $Y_{21}\ to\ Y_{23}$ in Population III). In Population IV, every estimator carried a bias of roughly 200 to 270, so MSE there is driven mainly by variance. The proposed estimators are not uniformly the least biased: in Population II, $t_{12(2)}$ had a bias of 1.26, compared with 2.7 to 4.5 for the proposed estimators. MSE and PRE are therefore the more defensible basis for the superiority claim. The efficiency gains were smallest in Population IV, suggesting that the benefit depends on the structure of the population and the relationship between the study and auxiliary variables.

CONCLUSION

Tables 5-9 28 present the comparative performance of the proposed variance estimators and existing variance estimators under five different populations. The evaluation was based on Bias, Mean Squared Error (MSE), and Percentage Relative Efficiency (PRE), where lower MSE and higher PRE indicate superior estimator performance. Across all five populations, the conventional sample variance estimator consistently exhibited the largest MSE values and the lowest efficiency, thereby serving as the baseline estimator with PRE equal to 100. This result confirms the inadequacy of the traditional variance estimator when auxiliary information is ignored. Among the existing estimators, the estimators proposed by Isaki (1983), Singh et al. (2011), and Yadav and Kadilar (2013) generally improved estimation efficiency relative to the sample variance estimator. In most cases, the estimator of Yadav and Kadilar (2013) emerged as the most efficient among the traditional estimators, producing lower MSE values and higher PREs than the Isaki (1983) and Singh et al. (2011) estimators. Conversely, the estimator of Asghar et al. (2014) consistently recorded the highest MSE values among the existing estimators and often yielded PRE values below 100, indicating inferior performance. The proposed class of variance estimators consistently outperformed all competing estimators across the five populations. The proposed estimators produced considerably smaller MSE values and substantially higher PRE values than both the conventional and existing estimators. In Population I (Table 5), the best proposed estimator attained a PRE exceeding 11,000, while in Population II (Table 6) the highest PRE exceeded 9,200. Similarly, for Population III (Table 7), the proposed estimators achieved PRE values above 20,000, representing the largest efficiency gains observed in the study. In Population IV (Table 8), PRE values exceeded 1,200, while in Population V (Table 9), the best proposed estimator attained a PRE greater than 20,800. A further examination of the bias values revealed that the proposed estimators generally maintained relatively small biases compared with most existing estimators. This indicates that the proposed estimators are not only more precise, as evidenced by their smaller MSEs, but are also more accurate in estimating the population variance.

Future research should extend the proposed variance estimators to more complex sampling designs such as stratified, systematic, cluster, multistage, and probability proportional to size sampling to assess their performance under practical survey conditions. Their robustness should also be investigated in the presence of non-response, measurement errors, missing data, and outliers, as these are common features of real-world surveys. In addition, the proposed methodology could be generalized to incorporate multiple auxiliary variables and adapted within calibration and model-assisted estimation frameworks to further improve efficiency. Further studies may also explore the application of the estimators to large-scale administrative and big data sources, as well as investigate the integration of machine learning techniques to model complex relationships between study and auxiliary variables. Finally, the estimators should be validated using real-life datasets from diverse fields such as agriculture, health, economics, environmental science, and official statistics, and extended to estimate other finite population parameters, including totals, ratios, coefficients of variation, quantiles, and inequality measures. Such extensions would broaden the applicability, robustness, and practical relevance of the proposed estimators in modern survey sampling.

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