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A Diagonally Implicit Modified Extended 2-Point Super Class of Block Backward Differentiation Formula for Solving Stiff Initial Value Problems

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ABSTRACT

This work develops a diagonally implicit modified extended 2-point super-class block backward differentiation formula (DIME2SBBDF) for stiff initial value problems (IVPs). The scheme is constructed by inserting a diagonal matrix into the coefficient matrix of the existing modified extended 2-point super-class block backward differentiation formula (ME2SBBDF). Two solution values are computed per integration step. For admissible values of the free $\rho \in (-1, 1)$, the method is consistent and third-order accurate. When $\rho = -\frac{1}{5}$, zero-stability and A-stability are verified. Convergence is also established theoretically. The method was implemented in C and tested on linear and nonlinear stiff IVPs. Comparisons with 1BDF, 2BBDF, and ME2SBBDF show that DIME2SBBDF generally gives better accuracy. The proposed method is therefore a viable alternative for stiff IVPs.

INTRODUCTION

Differential equations are widely used to model physical, biological, chemical, and engineering processes. They are usually classified as ordinary differential equations (ODEs) or partial differential equations (PDEs), depending on the number of independent variables. An ODE may be stiff or non-stiff depending on the behaviour of its solution and the stability requirements placed on the numerical method used to solve it.

An initial value problem (IVP) specifies the unknown function and, where required, its derivatives at a single initial point. A boundary value problem (BVP), by contrast, imposes conditions at two or more points. This study is concerned with the numerical solution of stiff IVPs using a diagonally implicit block method.

Analytical techniques can solve differential equations exactly when closed-form solutions exist. However, many practical differential equations cannot be solved analytically, so numerical methods are needed to approximate solutions at discrete points. A general first-order IVP is given by

$$y'(x) = f(x, y), \quad y(a) = y_0, \quad a \le x \le b \qquad (1)$$

Stiff ODE systems occur in many fields, including mathematical biology, chemical kinetics, electrical circuit analysis, and fluid dynamics. Such systems contain components that evolve on widely different time scales: some change rapidly while others change slowly. Explicit methods, such as explicit Runge–Kutta and Adams–Bashforth schemes, often require extremely small step sizes to remain stable, which increases computational cost.

Implicit methods are usually more suitable for stiff problems because they can use larger step sizes without losing stability. Among implicit linear multistep methods, the backward differentiation formula (BDF) introduced by Curtiss and Hirschfelder (1952) has been widely used for stiff IVPs because of its favourable stability properties.

Conventional BDF methods compute solutions sequentially, which can limit efficiency. Block backward differentiation formula (BBDF) methods address this by computing several solution values simultaneously per step. This reduces sequential computation and can improve efficiency. Several BBDF methods and their super-class formulations have been developed for stiff IVPs (Ibrahim et al., 2007; Zawawi et al., 2012; Suleiman et al., 2014; Musa, 2015; Babangida and Musa, 2016; Sagir et al., 2022; Alhassan et al., 2024).

Modified super-class BBDF formulations have also been reported (Musa et al., 2011, 2012, 2015; Abdullahi and Musa, 2021a, 2021b; Yusuf et al., 2024, 2025, 2026; Musa and Jaafar, 2026). These formulations introduce additional future points or free parameters into existing schemes, giving greater flexibility in accuracy and stability. A suitable choice of the free parameter can produce different members of the super class and may improve the resulting method.

In this study, a diagonally implicit modified extended 2-point super-class block backward differentiation formula (DIME2SBBDF) is developed. It is obtained by introducing a diagonal matrix into the coefficient matrix of the ME2SBBDF developed by Musa and Jaafar (2026). The method computes two solution values simultaneously per step. Its consistency, order, zero-stability, A-stability, and convergence are analysed. The method is implemented in C and applied to select linear and nonlinear stiff IVPs. Its performance is compared with 1BDF, 2BBDF, and ME2SBBDF.

MATERIALS AND METHODS

The starting point is the method of Musa and Jaafar (2026):

$$\sum_{j=0}^{3} \alpha_{j,i} y_{n+j-1} = \mathrm{h}\beta_{k,i}\,(f_{n+k} + \rho f_{n+k-2}), \qquad k = i = 1, 2 \tag{2}$$

where $\beta_{k-2,i} \neq 0$ i.e. $\beta_{k-2,i} = \rho\beta_{k,i}$ and $\rho \in (-1, 1)$. The method is suitable for stiff initial value problems and is A-stable for appropriate ρ in $(-1, 1)$.

To obtain a diagonally implicit 2-point super class formulation, Eq. (2) is modified as follows:

$$\sum_{j=0}^{1+k} \alpha_{j,i} y_{n+j-1} = \mathrm{h}\,\beta_{k,i}(f_{n+k} + \rho f_{n+\mathrm{k}-2})\ \mathrm{k} = \mathrm{i} = 1, 2 \tag{3}$$

The derivation uses Taylor series expansions. A linear difference operator for the first and second points is defined by

$$L_i\,[y\,(x_n, h)] = \sum_{j=0}^{2} \alpha_{j,i} y_{n+j-1} - h\,\beta_{k,i}(f_{n+k} + \rho f_{n+k-2}) = 0 \quad k = \mathrm{i} = 1,2 \tag{4}$$

Expanding Eq. (4) gives

$$L_i\,[y\,(x_n, h)] = \alpha_{0,i} y_{n-1} + \alpha_{1,i} y_n + \alpha_{2,i} y_{n+1} + \alpha_{3,i} y_{n+2} - h\,\beta_{k,i}(f_{n+k} + \rho f_{n+k-2}) = 0, k = i = 1,2 \tag{5}$$

First point

For the first point, set $k = \mathrm{i} = 1$ in Eq. (5), giving

$$L_i\,[y\,(x_n, h)] = \alpha_{0,1} y_{n-1} + \alpha_{1,1} y_n + \alpha_{2,1} y_{n+1} - h\,\beta_{k,1}(f_{n+k} + \rho f_{n+k-2}) = 0 \tag{6}$$

The corresponding continuous relation is

$$L_i\,[y\,(x_n, h)] = \alpha_{0,1} y(x_n - h) + \alpha_{1,1} y(x_n) + \alpha_{2,1}\,y(x_n + h) - h\beta_{1,1}[y'(x_n + h) + \rho y'(x_n - h)] = 0 \tag{7}$$

Expanding (7) as Taylor's series about x_n and collecting like terms gives

$$C_{0,1} y\,(x_n) + C_{1,1}\,h y'(x_n) + C_{2,1}\,h^2 y''(x_n) + C_{3,1}\,h^3 y'''\,(x_n) + \cdots \tag{8}$$

where

$$\left.\begin{aligned} C_{0,1} &= \alpha_{0,1} + \alpha_{1,1} + \alpha_{2,1} = 0; \\ C_{1,1} &= -\alpha_{0,1} + \alpha_{2,1} - \rho\beta_{1,1} - \beta_{1,1} = 0; \\ C_{2,1} &= \tfrac{1}{2}\alpha_{0,1} + \tfrac{1}{2}\alpha_{2,1} + \rho\beta_{1,1} - \beta_{1,1} = 0 \end{aligned}\right\} \tag{9}$$

Normalizing the coefficient of y_{n+1} to unity, i.e, $\alpha_{2,1} = 1$, and solving Eq. (9) simultaneously gives the required coefficients. Substitution into Eq. (7) yields

$$y_{n+1} = \frac{-3\rho+1}{\rho-3} y_{n-1} + \frac{4(\rho-1)}{\rho-3} y_n - \frac{2h\rho}{\rho-3} f_{n-1} - \frac{2h}{\rho-3} f_{n+1} \tag{10}$$

Second point

For the second point, set $k = \text{i} = 2$ in Eq. (5). Following the same procedure gives

$$y_{n+2} = \frac{-2(\rho+1)}{\rho-11} y_{n-1} - \frac{3(\rho-3)}{\rho-11} y_n + \frac{6(\rho-3)}{\rho-11} y_{n+1} - \frac{6h\rho}{\rho-11} f_n - \frac{6h}{\rho-11} f_{n+2} \quad (11)$$

Combining Eqs. (10) and (11), gives the proposed DIME2SBBDF method:

$$\left.\begin{aligned} y_{n+1} &= \frac{-3\rho+1}{\rho-3} y_{n-1} + \frac{4(\rho-1)}{\rho-3} y_n - \frac{2h\rho}{\rho-3} f_{n-1} - \frac{2h}{\rho-3} f_{n+1} \\ y_{n+2} &= \frac{-2(\rho+1)}{\rho-11} y_{n-1} - \frac{3(\rho-3)}{\rho-11} y_n + \frac{6(\rho-3)}{\rho-11} y_{n+1} - \frac{6h\rho}{\rho-11} f_n - \frac{6h}{\rho-11} f_{n+2} \end{aligned}\right\} \quad (12)$$

The stability properties of the super class of BDF and BBDF methods were studied by Musa and Jaafar (2026). Based on those stability requirements, the parameter is restricted to $\rho\epsilon(-1, 1)$.

For $\rho = -\frac{1}{5}$, substitution into Eq. (12) gives the following particular member:

$$\left.\begin{aligned} y_{n+1} &= -\frac{1}{2} y_{n-1} + \frac{3}{2} y_n - \frac{1}{8} h f_{n-1} + \frac{5}{8} h f_{n+1} \\ y_{n+2} &= \frac{1}{7} y_{n-1} - \frac{6}{7} y_n + \frac{12}{7} y_{n+1} - \frac{3}{28} h f_n + \frac{15}{28} h f_{n+2} \end{aligned}\right\} \quad (13)$$

The error constant is $E_4 = \begin{pmatrix} -\frac{46}{21} \\ -\frac{140}{27} \end{pmatrix} \neq 0$, which shows that the method is third order accurate.

Stability Analysis of the Method

For $\rho = -\frac{1}{5}$, the proposed method is expressed in matrix form as

$$\begin{pmatrix} 1 & 0 \\ -\frac{12}{7} & 1 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{7} & -\frac{6}{7} \end{pmatrix} \begin{pmatrix} y_{n-1} \\ y_n \end{pmatrix} + \text{h} \begin{pmatrix} -\frac{1}{8} & 0 \\ 0 & -\frac{3}{28} \end{pmatrix} \begin{pmatrix} f_{n-1} \\ f_n \end{pmatrix} + \text{h} \begin{pmatrix} \frac{5}{8} & 0 \\ 0 & \frac{15}{28} \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_{n+2} \end{pmatrix} \quad (14)$$

The coefficient matrix on the left-hand side is lower triangular, the method is diagonally implicit.

Equation (14) can be written compactly as

$$C_0 Y_m = C_1 Y_{m-1} + h\,(G_0\, F_m + G_1\, F_{m-1}) \quad (15)$$

where

$$C_0 = \begin{pmatrix} 1 & 0 \\ -\frac{12}{7} & 1 \end{pmatrix},\ C_1 = \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{7} & -\frac{6}{7} \end{pmatrix},\ G_0 = \begin{pmatrix} -\frac{1}{8} & 0 \\ 0 & -\frac{3}{28} \end{pmatrix},\ G_1 = \begin{pmatrix} \frac{5}{8} & 0 \\ 0 & \frac{15}{28} \end{pmatrix}$$

To examine the stability, consider the scalar test equation

$$y' = \lambda y, \quad (\lambda < 0 \text{ and } \lambda \text{ is complex}) \quad (16)$$

Substituting Eq. (16) into Eq. (15) and setting $\lambda h = \bar{h}$ gives

$$C_0 Y_m = C_1 Y_{m-1} + \bar{h}\,(G_0\, F_m + G_1\, F_{m-1}) \quad (17)$$

The stability polynomial is therefore

$$\text{Det}[(C_0 - \bar{h}G_1)\, t - (C_1 + \bar{h}G_0)] = 0 \quad (18)$$

Evaluating the determinant gives

$$R(t, \bar{h}) = t^2 - \frac{65}{56} t^2 \bar{h} - \frac{17}{14} t - \frac{4}{7} t\bar{h} + \frac{75}{224} t^2 \bar{h}^2 - \frac{15}{112} t\bar{h}^2 + \frac{3}{14} + \frac{9}{56} \bar{h} + \frac{3}{224} \bar{h}^2 = 0 \quad (19)$$

Setting $\bar{h} = 0$, gives the first characteristic polynomial,

$$t^2 - \frac{17}{14} \text{t} + \frac{3}{14} = 0 \quad (20)$$

Its roots are

$t = 0.2142857143$ and t = 1

Since one root is unity and the other lies strictly inside the unit circle, the root condition is satisfied. Therefore, the DIME2SBBDF method is zero-stable.

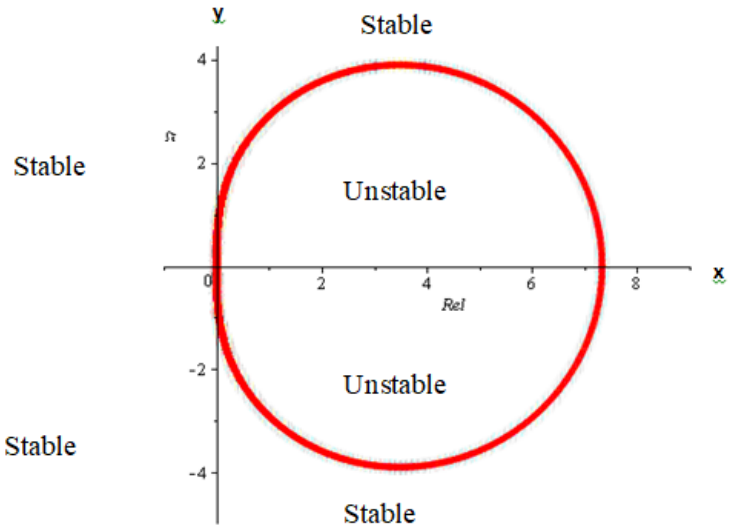


Figure 1; **Stability region of the DIME2SBBDF Method for $\rho = -\frac{1}{5}$**

The stability region for $\rho = -\frac{1}{5}$ is shown in **Figure 1.** The Stable region is the exterior of the plotted curve and contains the entire left half of the complex plane. Hence, the method is A-stable, consistent with Lambert (1993). This property is desirable for stiff IVPs.

Implementation of the Method

Newton's iteration is used to solve the nonlinear equations arising from the diagonally implicit formulation. Consider the stiff IVP

$$y' = f(x,y), y(x_0) = y_0 \quad x \in (a,b) \qquad (21)$$

Let y_i and $y(x_i)$ denote the numerical and exact solutions, respectively. The error at the $i-th$ step is

$$(error_i)_t = |(y_i)_t - (y(x_i))_t| \qquad (22)$$

The maximum absolute error is

$$\text{MAXE} = \underbrace{max}_{1\le i\le T}\left(\underbrace{max(error_i)_t}_{1\le i\le N}\right) \qquad (23)$$

where T denotes the total number of integration steps and N is the number of equations, following Ibrahim et al. (2007)

For $\rho = -\frac{1}{5}$, Eq. (14) can be written in the form;

$$\left.\begin{aligned} F_1 &= y_{n+1} + \tfrac{1}{8}hf_{n-1} - \tfrac{5}{8}hf_{n+1} - \varepsilon_1 \\ F_2 &= -\tfrac{12}{7}y_{n+1} + y_{n+2} + \tfrac{3}{28}hf_n - \tfrac{15}{28}hf_{n+2} - \varepsilon_2 \end{aligned}\right\} \qquad (24)$$

where the back values are

$$\left.\begin{aligned} \varepsilon_1 &= -\tfrac{1}{2}y_{n-1} + \tfrac{3}{2}y_n \\ \varepsilon_2 &= \tfrac{1}{7}y_{n-1} - \tfrac{6}{7}y_n \end{aligned}\right\} \qquad (25)$$

Let $y_{n+1}^{(i+1)}$ denote the $(i+1)^{th}$ iteration. The corresponding iteration error is defined by

Let, $y_{n+1}^{(i+1)}$ denote the $(i+1)^{th}$ iteration error is

$$e_{n+j}^{(i+1)} = y_{n+j}^{(i+1)} - y_{n+j}^{(i)}, \quad j = 1,2 \qquad (26)$$

Applying Newton's iteration to the 2-point diagonally implicit SBBDF method with $\rho = -\frac{1}{5}$ gives

$$e_{n+j}^{(i+1)} = -\left[\left[F'_j\left(y_{n+j}^{(i)}\right)\right]^{-1} F_j\left(y_{n+j}^{(i)}\right)\right], \quad j = 1,2 \qquad (27)$$

Equivalently,

$$\left[F'_j\left(y_{n+j}^{(i)}\right)\right] e_{n+j}^{(i+1)} = -\left[F_j\left(y_{n+j}^{(i)}\right)\right], \quad j = 1,2 \qquad (28)$$

For the 2-point formulation, this becomes

$$\left[F'_j\left(y_{n+1,n+2}^{(i)}\right)\right] e_{n+1,n+2}^{(i+1)} = -\left[F_j\left(y_{n+1,n+2}^{(i)}\right)\right], \qquad (29)$$

In matrix form, Eq. (29) becomes

$$\begin{pmatrix} 1-\frac{5}{8}h\frac{\delta f_{n+1}}{\delta y_{n+1}} & 0 \\ -\frac{12}{7} & 1-\frac{15}{28}h\frac{\delta f_{n+2}}{\delta y_{n+2}} \end{pmatrix}\begin{pmatrix} e_{n+1}^{(i+1)} \\ e_{n+2}^{(i+1)} \end{pmatrix}=\begin{pmatrix} -1 & 0 \\ \frac{12}{7} & -1 \end{pmatrix}\begin{pmatrix} y_{n+1}^{(i)} \\ y_{n+2}^{(i)} \end{pmatrix}+h\begin{pmatrix} -\frac{1}{8} & 0 \\ 0 & -\frac{3}{28} \end{pmatrix}\begin{pmatrix} f_{n-1}^{(i)} \\ f_{n}^{(i)} \end{pmatrix}+$$
$$h\begin{pmatrix} \frac{5}{8} & 0 \\ 0 & \frac{15}{28} \end{pmatrix}\begin{pmatrix} f_{n+1}^{(i)} \\ f_{n+2}^{(i)} \end{pmatrix}+\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} \quad (30)$$

A C program was written to implement Eq. (30) and obtain numerical solutions for the selected stiff IVPs. The derived formulae are applied in predictor–corrector mode, denoted PECE. Here P is predictor, E is function evaluation, C is corrector, and E is another function evaluation.

The approximate calculations for y_{n+1} and y_{n+2} , in PECE are:

(i) P (predict); $y_{n+1,n+2}^{(p)}$

(ii) E (evaluate); $y'_{n+1,n+2} = \mathrm{f}(x_{n+1,n+2}, y_{n+1,n+2}^{(p)})$

(iii) C (correct); $y_{n+1,n+2}^{(c)}$

(iv) E (evaluate); $y'_{n+1,n+2} = \mathrm{f}(x_{n+1,n+2}, y_{n+1,n+2}^{(c)})$

The approximations for $y_{n+1,n+2}^{(c)}$, a two-stage Newton iteration is used. For DIME2SBBDF when $\rho = -\frac{1}{5}$, the process are as follows.

(i) Compute the values for $e_{n+1,n+2}^{(i+1)} = A^{-1}B$, where;

A = $\begin{pmatrix} 1-\frac{5}{8}h\frac{\delta f_{n+1}}{\delta y_{n+1}} & 0 \\ -\frac{12}{7} & 1-\frac{15}{28}h\frac{\delta f_{n+2}}{\delta y_{n+2}} \end{pmatrix}$

B=

$$\begin{pmatrix} -y_{n+1}^{(i)}+\frac{5}{8}f_{n+1}^{(i)}-\frac{1}{8}f_{n-1}^{(i)}+\varepsilon_1 & 0 \\ \frac{12}{7}y_{n+1}^{(i)}-\frac{3}{28}f_n^{(i)} & -y_{n+2}^{(i)}+\frac{15}{28}f_{n+2}^{(i)}+\varepsilon_2 \end{pmatrix}$$

(ii) Calculate the corrected value for $y_{n+1,n+2}^{(i+1)}$ with $e_{n+1,n+2}^{(i+1)}$ from step (i).

(iii) Solve $e_{n+1,n+2}^{(i+1)} = A^{-1}B$ for second stage iteration.

(iv) The final value for $y_{n+1,n+2}^{(i+1)}$ are obtained from second stage iteration of $e_{n+1,n+2}^{(i+1)}$.

RESULTS AND DISCUSSION

Four stiff first-order IVPs are used to test DIME2SBBDF. They include linear and nonlinear cases, and the free parameter is set to $\rho = -\frac{1}{5}$.

Problem 1: Consider the stiff initial value problem

$$y' = -10xy, \qquad y(0) = 1, \qquad 0 \le x \le 10,$$

Exact Solution:

$$y(x) = e^{-5x^2}$$

Source: Suleiman *et al.* (2012).

Problem 2: Consider the system of stiff initial value problem

$$\begin{aligned} y'_1 &= -100y_1 + 9.901y_2, & y_1(0) &= 1 & 0 \le x \le 10 \\ y'_2 &= 0.1y_1 - y_2, & y_2(0) &= 10 \end{aligned}$$

Exact Solution:

$$y_1(x) = e^{-0.99x}$$
$$y_2(x) = 10e^{-0.99x}$$

Source: Musa et al. (2012)

Problem 3: Consider the system of stiff initial value problem

$$\begin{aligned} y'_1 &= -198y_1 + 199y_2, & y_1(0) &= 1 & 0 \le x \le 10 \\ y'_2 &= -398y_1 - 399y_2 & y_2(0) &= -1 \end{aligned}$$

Exact Solution:

$$y_1(x) = e^{-x}$$
$$y_2(x) = -e^{-x}$$

Source: Ibrahim et al. (2007)

Problem 4: Consider the stiff initial value problem

$$y' = -20y + 20sinx + cosx \quad y(0) = 1 \quad 0 \le x \le 2$$

Exact Solution;

$$y(x) = sinx + e^{-20x}$$

Source: Curtiss & Hirschfelder (1952)

These problems assess accuracy, stability, and computational efficiency. The DIME2SBBDF results are compared with 1BDF, 2BBDF, and ME2SBBDF. For each method, the maximum error (MAXE), number of steps (NS), and computational time (TIME) are recorded.

The notation used in Tables 1–4 is:

- 1BDF:1-Point Backward Differentiation Formula
- 2BBDF: 2Point Block Backward Differentiation Formula
- ME2SBBDF: Modified Extended 2Point Super Class of Block Backward Differentiation Formula.
- DIME2SBBDF: Diagonally Implicit Modify Extended 2-Point Super Class of Block Backward Differentiation Formula.
- h: Step size.
- NS: Total Number of Steps.
- MAXE: Maximum Absolute Error.
- Time: Computational Time in Seconds

Table 1. Numerical Results for Problem 1

h	Method	NS	MAXE	TIME
10^{-2}	1BDF	1000	$1\cdot 32240e-002$	$9\cdot 84800e-002$
	2BBDF	500	$2\cdot 47600e-002$	$6\cdot 49500e-002$
	ME2SBBDF	500	$3\cdot 79319e-003$	$7\cdot 56400e-002$
	DIME2SBBDF	500	$3\cdot 51573e-003$	$2\cdot 18800e-002$
10^{-3}	1BDF	10000	$1\cdot 44154e-003$	$8\cdot 57100e-002$
	2BBDF	5000	$2\cdot 86614e-003$	$7\cdot 27000e-002$
	ME2SBBDF	5000	$2\cdot 74409e-004$	$9\cdot 15700e-002$
	DIME2SBBDF	5000	$2\cdot 28705e-004$	$2\cdot 10800e-002$
10^{-4}	1BDF	100000	$1\cdot 45345e-004$	$9\cdot 56800e-002$
	2BBDF	50000	$2\cdot 90520e-004$	$7\cdot 91100e-002$
	ME2SBBDF	50000	$2\cdot 63842e-005$	$1\cdot 25000e-001$
	DIME2SBBDF	50000	$2\cdot 16290e-005$	$4\cdot 89900e-002$
10^{-5}	1BDF	1000000	$1\cdot 45464e-005$	$2\cdot 09400e-001$
	2BBDF	500000	$2\cdot 90911e-005$	$1\cdot 70900e-005$
	ME2SBBDF	500000	$2\cdot 62785e-006$	$4\cdot 88600e-001$
	DIME2SBBDF	500000	$2\cdot 15048e-006$	$2\cdot 32600e-001$
10^{-6}	1BDF	10000000	$1\cdot 45474e-006$	$1\cdot 29900e+000$
	2BBDF	5000000	$2\cdot 90951e-006$	$1\cdot 15900e+000$
	ME2SBBDF	5000000	$2\cdot 62684e-007$	$3\cdot 92000e+000$
	DIME2SBBDF	5000000	$2\cdot 14914e-007$	$1\cdot 96000e+000$

These problems assess accuracy, stability, and computational efficiency. The DIME2SBBDF results are compared with 1BDF, 2BBDF, and ME2SBBDF. For each method, the maximum error (MAXE), number of steps (NS), and computational time (TIME) are recorded.

Table 2. Numerical Results for Problem 2

h	Method	NS	MAXE	TIME
10^{-2}	1BDF	1000	$3\cdot 59052e-002$	$9\cdot 63500e-002$
	2BBDF	500	$7\cdot 11312e-002$	$6\cdot 78000e-002$
	ME2SBBDF	500	$7\cdot 18633e-003$	$9\cdot 29400e-002$
	DIME2SBBDF	500	$6\cdot 11124e-003$	$1\cdot 97600e-002$
10^{-3}	1BDF	10000	$3\cdot 63689e-003$	$7\cdot 76100e-002$
	2BBDF	5000	$7\cdot 26689e-003$	$6\cdot 89800e-002$
	ME2SBBDF	5000	$6\cdot 63511e-004$	$9\cdot 44400e-002$
	DIME2SBBDF	5000	$5\cdot 45082e-004$	$2\cdot 80800e-002$
10^{-4}	1BDF	100000	$3\cdot 64150e-004$	$1\cdot 12900e-001$

	2BBDF	50000	$7\cdot 28230e-004$	$9\cdot 54100e-002$
	ME2SBBDF	50000	$6\cdot 58175e-005$	$2\cdot 00400e-001$
	DIME2SBBDF	50000	$5\cdot 38727e-005$	$1\cdot 04600e-001$
10^{-5}	1BDF	1000000	$3\cdot 64196e-005$	$2\cdot 99100e-001$
	2BBDF	500000	$7\cdot 28384e-005$	$3\cdot 05500e-001$
	ME2SBBDF	500000	$6\cdot 57641e-006$	$1\cdot 29400e+000$
	DIME2SBBDF	500000	$5\cdot 38095e-006$	$6\cdot 80000e-001$
10^{-6}	1BDF	10000000	$3\cdot 64219e-006$	$2\cdot 34100e+000$
	2BBDF	5000000	$7\cdot 28384e-006$	$2\cdot 32500e+000$
	ME2SBBDF	5000000	$6\cdot 57465e-007$	$1\cdot 22200e+001$
	DIME2SBBDF	5000000	$5\cdot 38267e-007$	$5\cdot 91100e+000$

Table 2 shows that DIME2SBBDF maintains a favourable error profile over the range of step sizes considered. The proposed method produces accurate solutions while maintaining competitive computational cost.

Table 3. Numerical Results for Problem 3

h	Method	NS	MAXE	TIME
10^{-2}	1BDF	1000	$5\cdot 92215e+040$	$9\cdot 34500e-002$
	2BBDF	500	$7\cdot 18323e-003$	$6\cdot 19200e-002$
	ME2SBBDF	500	$7\cdot 26503e-004$	$7\cdot 25900e-002$
	DIME2SBBDF	500	$6\cdot 18072e-004$	$2\cdot 16600e-002$
10^{-3}	1BDF	10000	$3\cdot 67358e-004$	$8\cdot 11300e-002$
	2BBDF	5000	$7\cdot 34012e-004$	$7\cdot 39800e-002$
	ME2SBBDF	5000	$6\cdot 70274e-005$	$9\cdot 11300e-002$
	DIME2SBBDF	5000	$5\cdot 50660e-005$	$3\cdot 00100e-002$
10^{-4}	1BDF	100000	$3\cdot 67827e-005$	$9\cdot 199000e-002$
	2BBDF	50000	$7\cdot 35584e-005$	$8\cdot 16100e-002$
	ME2SBBDF	50000	$6\cdot 64830e-006$	$2\cdot 09000e-001$
	DIME2SBBDF	50000	$5\cdot 44176e-006$	$1\cdot 09700e-001$
10^{-5}	1BDF	1000000	$3\cdot 67875e-006$	$3\cdot 09900e-001$
	2BBDF	500000	$7\cdot 35741e-006$	$2\cdot 28980e-001$
	ME2SBBDF	500000	$6\cdot 64285e-007$	$1\cdot 34600e+000$
	DIME2SBBDF	500000	$5\cdot 43532e-007$	$6\cdot 64800e-001$
10^{-6}	1BDF	10000000	$3\cdot 67902e-007$	$2\cdot 33400e+000$
	2BBDF	5000000	$7\cdot 35742e-007$	$2\cdot 38900e+000$
	ME2SBBDF	5000000	$6\cdot 64135e-008$	$1\cdot 22700e+001$
	DIME2SBBDF	5000000	$5\cdot 43706e-008$	$5\cdot 86000e+000$

The numerical results for Problem 3 are presented in Table 3. As the step size decreases, the maximum error generally decreases, indicating that the numerical solution approaches the exact solution. DIME2SBBDF performs favourably compared with the other methods.

Table 4. Numerical Results for Problem 4

h	Method	NS	MAXE	TIME
10^{-2}	1BDF	199	$4\cdot 97463e-002$	$8\cdot 45000e-002$
	2BBDF	100	$7\cdot 82684e-002$	$6\cdot 51700e-002$
	ME2SBBDF	100	$3\cdot 92590e-002$	$7\cdot 33000e-002$
	DIME2SBBDF	100	$3\cdot 97267e-002$	$1\cdot 88100e-002$
10^{-3}	1BDF	2000	$7\cdot 14470e-003$	$5\cdot 83400e-002$
	2BBDF	1000	$1\cdot 40171e-002$	$6\cdot 34000e-002$
	ME2SBBDF	1000	$1\cdot 58555e-003$	$1\cdot 49000e-001$
	DIME2SBBDF	1000	$1\cdot 39669e-003$	$1\cdot 86900e-002$
10^{-4}	1BDF	20000	$7\cdot 33578e-004$	$6\cdot 89200e-002$
	2BBDF	10000	$1\cdot 46435e-003$	$7\cdot 28900e-002$
	ME2SBBDF	10000	$1\cdot 35256e-004$	$7\cdot 83100e-002$
	DIME2SBBDF	10000	$1\cdot 11571e-004$	$3\cdot 68700e-002$
10^{-5}	1BDF	199999	$7\cdot 35458e-005$	$1\cdot 83100e-001$
	2BBDF	100000	$1\cdot 47063e-004$	$1\cdot 27200e-001$
	ME2SBBDF	100000	$1\cdot 33070e-005$	$2\cdot 69200e-001$
	DIME2SBBDF	100000	$1\cdot 08965e-005$	$1\cdot 47900e-001$
10^{-6}	1BDF	2000000	$7\cdot 35646e-006$	$5\cdot 75000e-001$
	2BBDF	1000000	$1\cdot 47126e-005$	$5\cdot 48500e-001$
	ME2SBBDF	1000000	$1\cdot 32853e-006$	$1\cdot 68700e+001$
	DIME2SBBDF	1000000	$1\cdot 08707e-006$	$1\cdot 08100e+000$

Table 4 presents the numerical results for Problem 4. The proposed DIME2SBBDF method gives competitive accuracy and computational time across the different step sizes considered. The results further support the effectiveness of the proposed method for stiff IVPs.

For a clearer comparison of the accuracy of the four methods, the maximum errors are also presented graphically using MATLAB version R2007b (2007). Figures 2–5 show plots of the scaled log_{10} MAXE against $log_{10}\, h$ for Problems 1–4, respectively.

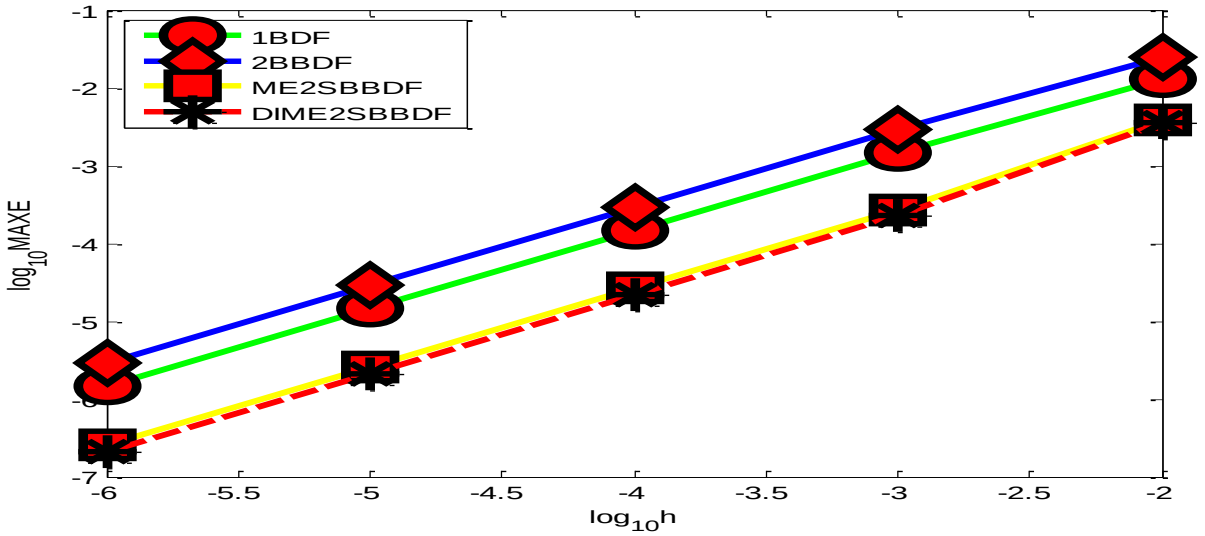


Figure 2: ***Graph of log_{10} MAXE against $log_{10}\, h$ for problem 1.***

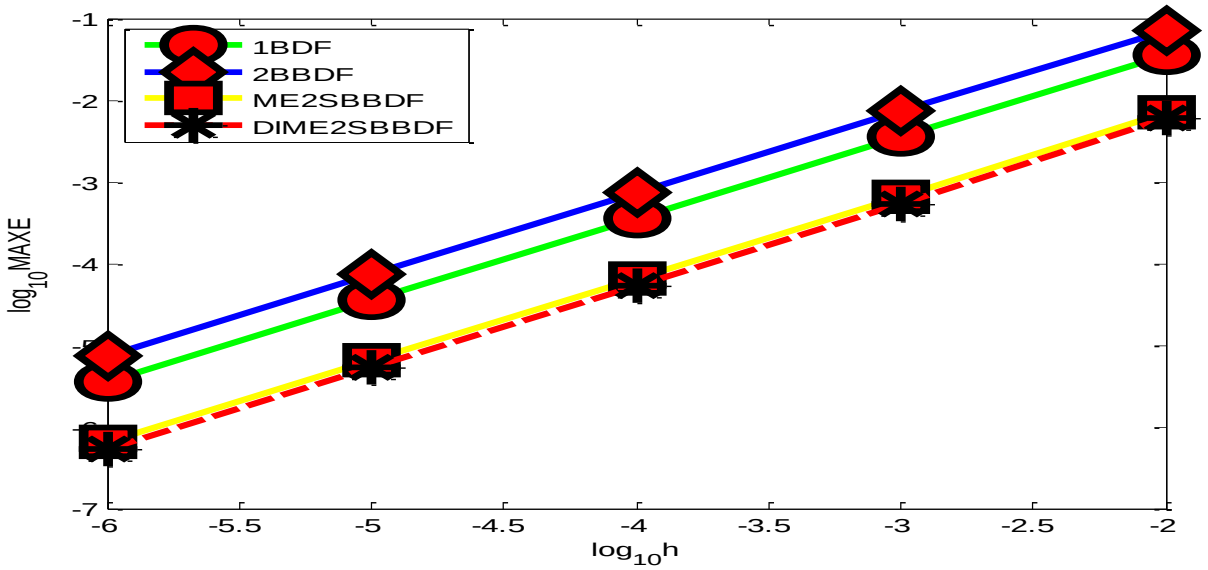


Figure 3: ***Graph of log_{10} MAXE against log_{10} h for problem 2.***

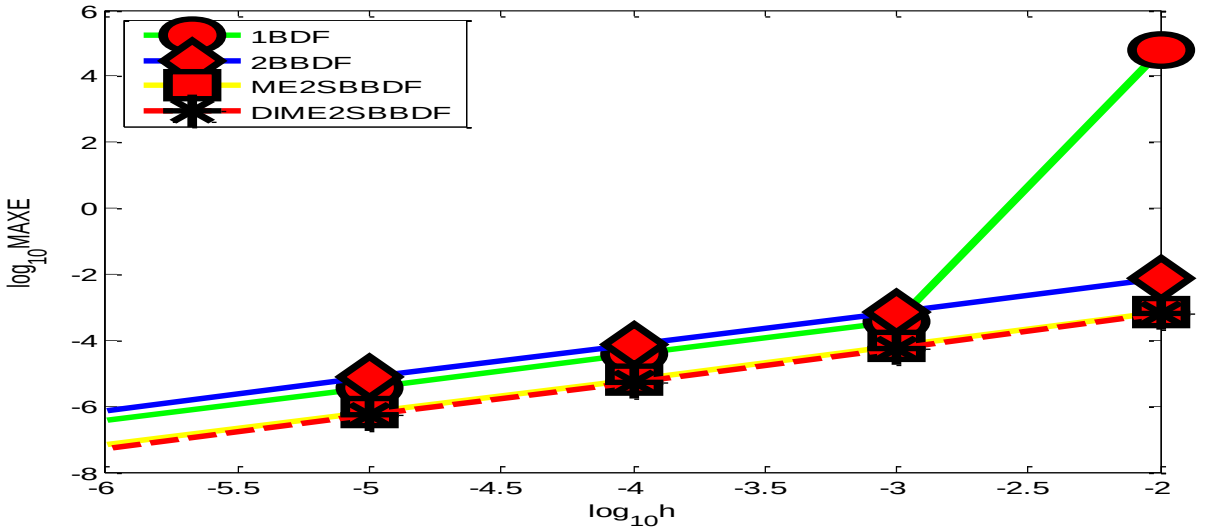


Figure 4: ***Graph of log_{10} MAXE against log_{10} h for problem 3.***

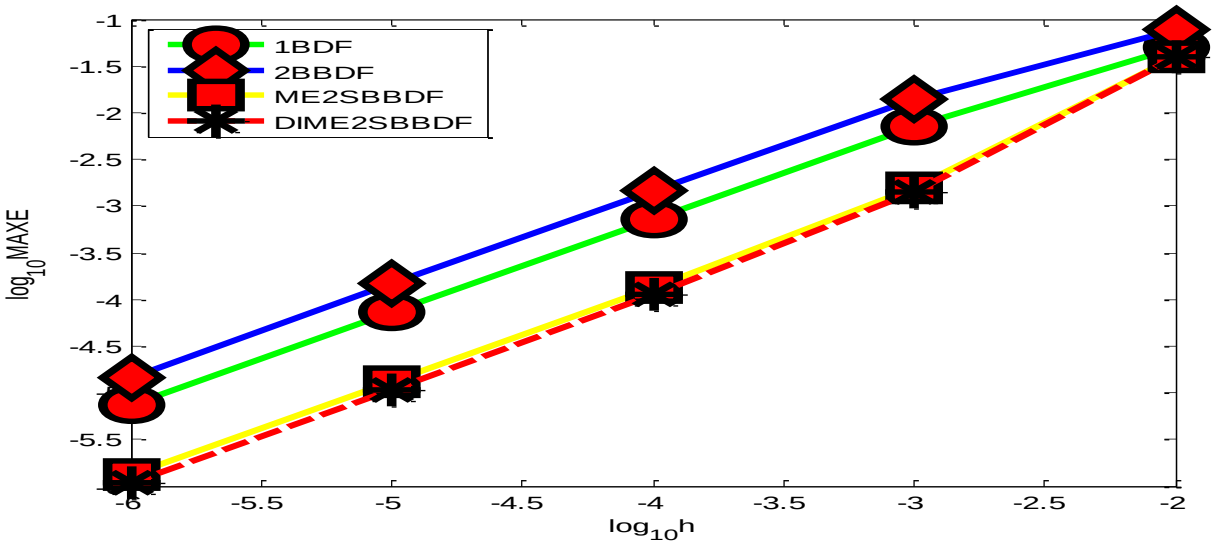


Figure 5: ***Graph of log_{10} MAXE against log_{10} h for problem 4.***

The results in Tables 1–4 and Figures 2–5 show that DIME2SBBDF generally produces smaller maximum errors than 1BDF, 2BBDF, and ME2SBBDF. This indicates improved accuracy for the test problems considered. Reducing the step size generally reduces the maximum error, which is consistent with the convergence behaviour expected of a consistent numerical method.

The computational time generally increases as the step size decreases because smaller step sizes require more integration steps. Despite this increase, the computational time of DIME2SBBDF remains competitive with the other methods. This suggests that computing two solution values per step does not compromise computational efficiency.

Overall, the numerical experiments demonstrate that DIME2SBBDF provides an effective combination of accuracy and computational efficiency for the stiff IVPs considered.

CONCLUSION

A new diagonally implicit modified extended 2-point super-class block backward differentiation formula (DIME2SBBDF) has been developed for stiff IVPs. The method computes two solution values simultaneously per integration step, giving a block formulation with computational advantages.

A free parameter is included in the formulation, allowing different members of the method to be obtained. Theoretical analysis shows that the method is consistent and third-order accurate. It is also zero-stable and A-stable for an appropriate parameter choice. Convergence follows from these properties.

The method was implemented in C and tested on four stiff IVPs. Results were compared with 1BDF, 2BBDF, and ME2SBBDF. DIME2SBBDF generally achieved smaller maximum errors while maintaining competitive computational times. The method is therefore accurate and efficient for the class of stiff IVPs considered and provides a useful alternative for linear and nonlinear stiff initial value problems.

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