Journal of Basics and Applied Sciences Research (JOBASR)
ISSN (print): 3026-9091, ISSN (online): 1597-9962
Volume 4(5) September 2026

DOI: https://dx.doi.org/10.4314/jobasr.v4i5.20

Convergence and Order of the Diagonally Implicit Extended 3-Point Super Class of Block Backward Differentiation Formula for Stiff Initial Value Problems

[1]*Yusuf H., [1,2]Musa H. & [3]Mohammad A.

[1*]Department of Mathematics, Faculty of Physical Sciences, Federal University Dutsin-Ma, Katsina State, Nigeria
[2]Department of Mathematics, Faculty of Natural and Applied Sciences, Umaru Musa Yar'adua University Katsina, Katsina State, Nigeria
[3]Department of Mathematics, Faculty of Computing and Mathematical Sciences, Aliko Dangote University of Science and Technology Wudil, Kano, Nigeria.
Correspondence E-mail: yhamza@fudutsinma.edu.ng:+2348062368886.

Keywords:

Convergence; consistency; zero-stability; order of accuracy; stiff initial value problems.

ABSTRACT

This study investigates the convergence properties of the diagonally implicit extended 3-point superclass block backward differentiation formula (DIE3SBBDF) developed by Yusuf et al. (2026) for the numerical solution of stiff initial value problems. The convergence of the method is established by analyzing its consistency and zero-stability properties. The order of accuracy of the method is also determined, and the method is shown to be of fifth order. These results provide a theoretical basis for the convergence and accuracy of the DIE3SBBDF method in solving first-order stiff initial value problems.

INTRODUCTION

Ordinary differential equations (ODEs) provide a mathematical framework for describing how quantities change over time. They are widely used in modeling physical, chemical, biological, and engineering processes, including chemical reactions, electrical circuits, fluid flow, vibration, and population dynamics (Lambert, 1973). Despite their broad applications, obtaining reliable numerical solutions to some differential equations remains challenging, particularly when the problems are stiff. Stiff systems often contain solution components that evolve at significantly different rates. As a result, a numerical method may need to use very small step sizes to maintain stability, even when the solution varies relatively smoothly. This requirement can increase computational costs and limit the efficiency of numerical simulations. Developing numerical methods that provide accurate and stable solutions without unnecessary computational effort therefore remains an important area of research.

The numerical treatment of stiff ODEs has attracted considerable attention since the early development of methods specifically designed to address their stability requirements. Curtiss and Hirschfelder (1952) examined the integration of stiff equations, contributing to the development of numerical approaches for this class of problems. Among the methods subsequently developed, backward differentiation formulae (BDFs) have become widely used because of their suitability for solving stiff initial value problems (IVPs). Further developments have extended the traditional BDF approach to block formulations, providing alternative ways to compute numerical solutions and analyze their accuracy and stability.

The general form of a system of first-order initial value problem (IVP) for an ordinary differential equation can be expressed as

$$u'_i = f_i(x,u_i),\ u_i(a)=\theta_i,\ x\in[a,b],\quad i=1,2,3,\ldots,n \qquad (1)$$

where u_i, u'_i, f_i and θ_i are n-dimensional vectors. When the IVP is stiff, numerical methods must satisfy demanding stability requirements to produce reliable approximations efficiently. This consideration motivates

the development and analysis of numerical methods specifically designed for stiff problems.

Numerical methods for solving stiff ODEs can broadly be classified into block and non-block methods. Block methods compute several approximate solution values simultaneously within an integration step. An example is the block backward differentiation formula (BBDF). In contrast, non-block methods compute the solution sequentially, with each new approximation obtained from previously computed values. The backward differentiation formula (BDF) is a well-known example of a non-block method.

Several non-block methods have been developed for the numerical solution of stiff initial value problems (Cash, 1980, 1999), while considerable research has also been devoted to block methods (Musa et al., 2011; Musa et al., 2012; Suleiman et al., 2014; Musa et al., 2015; Musa et al., 2016b; Musa and Unwala, 2019; Musa and Bala, 2019; Sagir et al., 2022; Alhassan et al., 2023, 2024; Yusuf et al., 2026).

The order, consistency, stability, and convergence of block backward differentiation formulae are essential properties for establishing the accuracy and reliability of numerical solutions to stiff initial value problems. Several studies have addressed these properties for different block formulations. Musa et al. (2012) investigated the convergence and order of a 3-point block extended backward differentiation formula. Musa (2013) studied the convergence and order of a 2-point improved block backward differentiation formula, while Musa et al. (2016a) investigated the convergence of a diagonally implicit superclass of block backward differentiation formulas. Abdullahi and Musa (2021) examined the order and convergence of an enhanced 3-point fully implicit superclass of block backward differentiation formulas for solving first-order stiff initial value problems.

More recently, Sagir et al. (2022) developed an A-stable block integrator for first-order systems of ordinary differential equations. Alhassan et al. (2023) presented a fifth-order variable-step-size block backward differentiation formula with off-step points for stiff ordinary differential equations. Alhassan et al. (2024) investigated the convergence and stability properties of a diagonally implicit 3-point block backward differentiation formula for first-order stiff initial value problems. Similarly, Yusuf et al. (2025) analysed the order and convergence of a new fixed-coefficient diagonally implicit block backward differentiation formula for the numerical solution of stiff ordinary differential equations.

Among the methods developed for stiff problems, Yusuf et al. (2026) introduced the diagonally implicit extended 3-point superclass block backward differentiation formula (DIE3SBBDF) for solving stiff initial value problems. The method computes three solution values concurrently at each integration step and is designed to provide an efficient numerical treatment of stiff systems.

Although the DIE3SBBDF method has been developed for the numerical solution of stiff initial value problems, establishing its theoretical properties is essential before its numerical performance can be fully assessed. In particular, the order, consistency, and zero-stability of the method must be established to determine its convergence.

Therefore, this study investigates the order and convergence properties of the DIE3SBBDF method developed by Yusuf et al. (2026). The order of the method is first determined, after which its consistency and zero-stability properties are analyzed. The results establish the convergence of the method for solving first-order stiff initial value problems.

MATRIALS AND METHOD

The DIE3SBBDF method developed by Yusuf et al. (2026) can be defined as follows:

$$\left.\begin{aligned}
u_{n+1} &= -\frac{2(\rho-1)}{\rho+11}u_{n-2} - \frac{3(\rho+3)}{\rho+11}u_{n-1} + \frac{6(\rho+3)}{\rho+11}u_n + \frac{6h}{\rho+11}f_{n+1} - \frac{6h\rho}{\rho+11}f_{n-1} \\
u_{n+2} &= \frac{(\rho-3)}{\rho+25}u_{n-2} - \frac{8(\rho-2)}{\rho+25}u_{n-1} - \frac{36}{\rho+25}u_n + \frac{8(\rho+6)}{\rho+25}u_{n+1} + \frac{12h}{\rho+25}f_{n+2} - \frac{12h\rho}{\rho+25}f_n \\
u_{n+3} &= -\frac{2(\rho-6)}{3\rho+137}u_{n-2} + \frac{15(\rho-5)}{3\rho+137}u_{n-1} - \frac{20(3\rho-10)}{3\rho+137}u_n + \frac{20(\rho-15)}{3\rho+137}u_{n+1} + \frac{30(3\rho+10)}{3\rho+137}u_{n+2} + \frac{60h}{3\rho+137}f_{n+3} - \frac{60h\rho}{3\rho+137}f_{n+1}
\end{aligned}\right\} \quad (2)$$

The formulation in Eq. (2) can be expressed in matrix form as

$$\left.\begin{pmatrix} 1 & 0 & 0 \\ -\frac{8(\rho+6)}{\rho+25} & 1 & 0 \\ -\frac{20(\rho-15)}{3\rho+137} & -\frac{30(3\rho+10)}{3\rho+137} & 1 \end{pmatrix}\begin{pmatrix} u_{n+1} \\ u_{n+2} \\ u_{n+3} \end{pmatrix} = \begin{pmatrix} -\frac{2(\rho-1)}{\rho+11} & -\frac{3(\rho+3)}{\rho+11} & \frac{6(\rho+3)}{\rho+11} \\ \frac{(\rho-3)}{\rho+25} & -\frac{8(\rho-2)}{\rho+25} & -\frac{36}{\rho+25} \\ -\frac{2(\rho-6)}{3\rho+137} & \frac{15(\rho-5)}{3\rho+137} & -\frac{20(3\rho-10)}{3\rho+137} \end{pmatrix}\begin{pmatrix} u_{n-2} \\ u_{n-1} \\ u_n \end{pmatrix} + h\begin{pmatrix} 0 & -\frac{6\rho}{\rho+11} & 0 \\ 0 & 0 & -\frac{12\rho}{\rho+25} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix} + h\begin{pmatrix} \frac{6}{\rho+11} & 0 & 0 \\ 0 & \frac{12}{\rho+25} & 0 \\ -\frac{60\rho}{3\rho+137} & 0 & \frac{60}{3\rho+137} \end{pmatrix}\begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{pmatrix}\right\} \quad (3)$$

The matrix formulation in Eq. (3) computes three solution values concurrently at each integration step. Further details concerning the derivation and formulation of the DIE3SBBDF method can be found in Yusuf et al. (2026).

Order of the Method

Definition 1. Linear multi-step method (LMM)

A general linear multistep method (LMM) can be written in the form

$$\sum_{j=0}^{k} \alpha_j u_{n+j} = h\sum_{j=0}^{k} \beta_j f_{n+j} \qquad (4)$$

where α_j and β_j are constants and $\alpha_j \neq 0$.But α_0 and β_0 cannot be zero at the same time. The coefficients α_j and β_j define the first and second characteristic polynomials associated with the method.

Definition 2. Order of the block method

The order of the block method given by Eq. (2) can be determined using the associated linear difference operator *L*, defined by

$$L\{u(x),h\} = \sum_{j=0}^{k}\left[A_j u(x+jh) - hB_j u'(x+jh)\right] \quad (5)$$

where $u(x)$ is an arbitrary sufficiently differentiable test function defined on [a, b] and A_j and B_j are constant column matrices.

Expanding the function $u(x+jh)$ and it's derivatives $u'(x+jh)$ as a Taylor's series about x_n gives;

$$u(x+jh) = u(x) + (jh)u'(x) + \frac{(jh)^2}{2!}u''(x) + \frac{(jh)^3}{3!}u'''(x) + \frac{(jh)^4}{4!}u^{iv}(x) + \ldots \quad (6)$$

$$u'(x+jh) = u'(x) + (jh)u''(x) + \frac{(jh)^2}{2!}u'''(x) + \frac{(jh)^3}{3!}u^{iv}(x) + \frac{(jh)^4}{4!}u^{v}(x) + \ldots \quad (7)$$

Substituting Eqs. (6) and (7) into Eq. (5) gives

$$\begin{aligned} L[y(x),h] = \sum_{j=0}^{5} A_j[u(x) + (jh)u'(x) + \frac{(jh)^2}{2!}u''(x) + \frac{(jh)^3}{3!}u'''(x) + \frac{(jh)^4}{4!}u^{iv}(x) + \ldots] \\ -h\sum_{j=0}^{5} B_j[u'(x) + (jh)u''(x) + \frac{(jh)^2}{2!}u'''(x) + \frac{(jh)^3}{3!}u^{iv}(x) + \frac{(jh)^4}{4!}u^{v}(x) + \ldots] \end{aligned} \quad (8)$$

and, after collecting the corresponding powers of *h*,

$$L[u(x),h] = \sum_{j=0}^{5} A_j[u(x)] + \sum_{j=0}^{5}[jA_j - D_j]hu'(x) + \frac{1}{2!}\sum_{j=0}^{5}[j^2A_j - 2jD_j]h^2u''(x)$$

$$+\frac{1}{3!}\sum_{j=0}^{5}[j^3A_j-3j^2D_j]h^3u'''(x)+\frac{1}{4!}\sum_{j=0}^{5}[j^4A_j-4j^3D_j]h^4u^{iv}(x)+\ldots \quad (9)$$

The block method in Eq. (2), together with its associated linear difference operator in Eq. (5), is said to be of order p if the coefficients satisfy the corresponding order conditions up to p.

To determine the order of the DIE3SBBDF method, the matrix equation in Eq. (3) is rearranged as follows:

$$\begin{pmatrix} \frac{2(\rho-1)}{\rho+11} & \frac{3(\rho+3)}{\rho+11} & -\frac{6(\rho+3)}{\rho+11} \\ -\frac{(\rho-3)}{\rho+25} & \frac{8(\rho-2)}{\rho+25} & \frac{36}{\rho+11} \\ \frac{2(\rho-6)}{3\rho+137} & -\frac{15(\rho-5)}{3\rho+137} & \frac{20(3\rho-10)}{3\rho+137} \end{pmatrix}\begin{pmatrix} u_{n-2} \\ u_{n-1} \\ u_n \end{pmatrix}+\begin{pmatrix} 1 & 0 & 0 \\ -\frac{8(\rho+6)}{\rho+25} & 1 & 0 \\ \frac{20(\rho-15)}{3\rho+137} & -\frac{30(\rho+10)}{3\rho+137} & 1 \end{pmatrix}\begin{pmatrix} u_{n+1} \\ u_{n+2} \\ u_{n+3} \end{pmatrix}=$$

$$h\begin{pmatrix} 0 & -\frac{6}{\rho+11} & 0 \\ 0 & 0 & -\frac{12\rho}{\rho+25} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}+h\begin{pmatrix} \frac{6}{\rho+11} & 0 & 0 \\ 0 & \frac{12}{\rho+25} & 0 \\ -\frac{60\rho}{3\rho+137} & 0 & \frac{60\rho}{3\rho+137} \end{pmatrix}\begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{pmatrix} \quad (10)$$

where the resulting coefficient matrices are

$$A_0=\begin{pmatrix} \frac{2(\rho-1)}{\rho+11} \\ -\frac{(\rho-3)}{\rho+25} \\ \frac{2(\rho-6)}{3\rho+137} \end{pmatrix}, A_1=\begin{pmatrix} \frac{3(\rho+3)}{\rho+11} \\ \frac{8(\rho-2)}{\rho+25} \\ -\frac{15(\rho-5)}{3\rho+137} \end{pmatrix}, A_2=\begin{pmatrix} -\frac{6(\rho+3)}{\rho+11} \\ \frac{36}{\rho+11} \\ \frac{20(3\rho-10)}{3\rho+137} \end{pmatrix}, A_3=\begin{pmatrix} 1 \\ -\frac{8(\rho+6)}{\rho+25} \\ \frac{20(\rho-15)}{3\rho+137} \end{pmatrix},$$

$$A_4=\begin{pmatrix} 0 \\ 1 \\ -\frac{30(\rho+10)}{3\rho+137} \end{pmatrix}, A_5=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, D_0=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, D_1=\begin{pmatrix} -\frac{6\rho}{\rho+11} \\ 0 \\ 0 \end{pmatrix}, D_2=\begin{pmatrix} 0 \\ -\frac{12\rho}{\rho+25} \\ 0 \end{pmatrix}, D_3=\begin{pmatrix} \frac{6\rho}{\rho+11} \\ 0 \\ -\frac{60\rho}{3\rho+137} \end{pmatrix}$$

$$D_4=\begin{pmatrix} 0 \\ \frac{12\rho}{\rho+25} \\ 0 \end{pmatrix}, D_5=\begin{pmatrix} 0 \\ 0 \\ \frac{60}{3\rho+137} \end{pmatrix}$$

Substitution of these coefficient matrices into the order conditions gives

$$E_0 = \sum_{j=0}^{5} A_j = A_0 + A_1 + A_2 + A_3 + A_4 + A_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$E_1 = \sum_{j=0}^{5} [\, jA_j - D_j \,] = [(0 + A_1 + 2A_2 + 3A_3 + 4A_4 + 5A_5) - (0 + D_1 + D_2 + D_3 + D_4 + D_5)] = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$E_2 = \sum_{j=0}^{5} [\frac{1}{2} j^2 A_j - jD_j] = [(0 + \frac{1}{2} A_1 + 2A_2 + \frac{9}{2} A_3 + 8A_4 + \frac{25}{2} A_5) - (0 + D_1 + 2D_2 + 3D_3 + 4D_4 + 5D_5)] = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$E_3 = \sum_{j=0}^{5} [\frac{1}{6} j^3 A_j - \frac{1}{2} j^2 D_j] = [(0 + \frac{1}{6} A_1 + \frac{4}{3} A_2 + \frac{9}{2} A_3 + \frac{32}{3} A_4 + \frac{125}{6} A_5) - (0 + \frac{1}{2} D_1 + 2D_2 + \frac{9}{2} D_3 + 8D_4 + \frac{25}{2} D_5)] = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$E_4 = \sum_{j=0}^{5} [\frac{1}{24} j^4 A_j - \frac{1}{6} j^3 D_j] = [(0 + \frac{1}{24} A_1 + \frac{2}{3} A_2 + \frac{27}{8} A_3 + \frac{32}{3} A_4 + \frac{625}{24} A_5) - (0 + \frac{1}{6} D_1 + \frac{4}{3} D_2 + \frac{9}{2} D_3 + \frac{32}{3} D_4 + \frac{125}{6} D_5)] = \begin{pmatrix} \frac{\rho - 3}{2(\rho + 11)} \\ 0 \\ 0 \end{pmatrix},$$

$$E_5 = \sum_{j=0}^{5} [\frac{1}{120} j^5 A_j - \frac{1}{24} j^4 D_j] = [(0 + \frac{1}{120} A_1 + \frac{4}{15} A_2 + \frac{81}{40} A_3 + \frac{128}{15} A_4 + \frac{625}{24} A_5) - (0 + \frac{1}{24} D_1 + \frac{2}{3} D_2 + \frac{27}{8} D_3 + \frac{32}{3} D_4 + \frac{625}{24} D_5)] = \begin{pmatrix} \frac{7\rho - 27}{10(\rho + 11)} \\ \frac{2\rho - 12}{5(\rho + 25)} \\ 0 \end{pmatrix},$$

$$E_6 = \sum_{j=0}^{5} [\frac{1}{720} j^6 A_j - \frac{1}{120} j^5 D_j] = [(0 + \frac{1}{720} A_1 + \frac{4}{45} A_2 + \frac{81}{40} A_3 + \frac{256}{45} A_4 + \frac{3125}{144} A_5) - (0 + \frac{1}{120} D_1 + \frac{4}{15} D_2 + \frac{81}{40} D_3 + \frac{128}{15} D_4 + \frac{625}{24} D_5)] = \begin{pmatrix} \frac{8\rho - 39}{15(\rho + 11)} \\ \frac{4(\rho - 7)}{5(\rho + 25)} \\ \frac{\rho - 10}{3\rho + 137} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

Thus, the first non-zero truncation-error coefficient occurs at the fifth-order term. Consequently, the DIE3SBBDF method is of order five with error constant $\left(\frac{8\rho - 39}{15(\rho + 11)} \quad \frac{4(\rho - 7)}{5(\rho + 25)} \quad \frac{\rho - 10}{3\rho + 137} \right)^T$ Hence, the method has fifth-order accuracy.

RESULTS AND DISCUSSION

The convergence of a numerical method is an essential property that must be established before the method can be regarded as theoretically reliable for solving an initial value problem.

For linear multistep methods, convergence is characterized by consistency and zero-stability. The fundamental convergence theorem for linear multistep methods states that a linear multistep method is convergent if and only if it is both consistent and zero-stable (Lambert, 1973).

Therefore, to establish the convergence of the DIE3SBBDF method, its zero-stability and consistency properties are considered separately.

Definition 3. Characteristic polynomial
The characteristic polynomials associated with the linear multistep method in Eq. (4) are defined by

$$\pi(\xi,\bar{h}) = \rho(\xi) - \bar{h}\sigma(\xi) \qquad (11)$$

where $\bar{h} = \lambda h$ and $\lambda = \frac{\delta f}{\delta y}$ $\rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j$ and $\sigma(\xi) = \sum_{j=0}^{k} \beta_j \xi^j$

The first and second characteristic polynomials are denoted by ρ(ξ) and σ(ξ) respectively.

Definition 4. Zero-stability
A linear multistep method is said to be zero-stable if all the roots of its first characteristic polynomial $\rho(\xi)$ satisfy $|\xi| \le 1$ and every root satisfying $|\xi| = 1$ is simple.

Definition 5. Linear Difference Operator
The linear difference operator associated with the linear multistep method is defined as

$$L\{u(x),h\} = \sum_{j=0}^{k}\left[A_j u(x+jh) - hB_j u'(x+jh)\right] \qquad (12)$$

where $u(x)$ is an arbitrary sufficiently differentiable test function defined on the interval [a, b].
Expanding $u(x+jh)$ and its derivatives as Taylor series about x_n, and collecting like terms gives

$$L\{u(x),h\} = C_0 u(x) + C_1 h u'(x) + C_2 h^2 u''(x) + \ldots + C_r h^r u^r(x) + \ldots \qquad (13)$$

where the coefficients are given by

$$\left.\begin{aligned} C_0 &= \alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_k \\ C_1 &= \alpha_1 + 2\alpha_2 + \cdots + k\alpha_k - (\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k) \\ &\vdots \\ C_r &= \tfrac{1}{r!}(\alpha_1 + 2^r\alpha_2 + \cdots + k^r\alpha_r) - \tfrac{1}{(r-1)!}(\beta_1 + 2^{r-1}\beta_2 + \cdots + k^{r-1}\beta_k) \\ & r = 2, 3, 4, \ldots \end{aligned}\right\} \qquad (14)$$

Definition 6. Consistency
A linear multistep method is said to be consistent if its order is at least one. Equivalently, the method is consistent if the first two order conditions are satisfied:

$$\sum_{j=0}^{k}\alpha_j = 0 \text{ and } \sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k}\beta_j \qquad (15)$$

where the quantities involved are the corresponding constant column matrices.

Stability formulation
Let U_m and F_m denote the corresponding solution and function-value vectors. The general k-block, and r-point method can be expressed in matrix form as

$$C_0 U_m = C_1 U_{m-1} + h(G_0 F_m + G_1 F_{m-1}) \qquad (16)$$

where, C_0, C_1, G_0 and G_1 are defined as above.

Then, $n = mr, m = 0,1, \ldots$ *and* $r = 3$

$$U_m = \begin{pmatrix} u_{n+1} \\ u_{n+2} \\ u_{n+3} \end{pmatrix} = \begin{pmatrix} u_{3m+1} \\ u_{3m+2} \\ u_{3m+3} \end{pmatrix}, U_{m-1} = \begin{pmatrix} u_{n-2} \\ u_{n-1} \\ u_n \end{pmatrix} = \begin{pmatrix} u_{3m-2} \\ u_{3m-1} \\ u_{3m} \end{pmatrix} = \begin{pmatrix} u_{3(m-1)+1} \\ u_{3(m-1)+2} \\ u_{3(m-1)+3} \end{pmatrix},$$

$$F_m = \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{pmatrix} = \begin{pmatrix} f_{3m+1} \\ f_{3m+2} \\ f_{3m+3} \end{pmatrix}, F_{m-1} = \begin{pmatrix} f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix} = \begin{pmatrix} f_{3m-2} \\ f_{3m-1} \\ f_{3m} \end{pmatrix} = \begin{pmatrix} f_{3(m-1)+1} \\ f_{3(m-1)+2} \\ f_{3(m-1)+3} \end{pmatrix},$$

The stability of the DIE3SBBDF method is analyzed using the standard linear test equation

$$u' = \lambda u \qquad (17)$$

where λ is a complex constant with $Re(\lambda) < 0$

Substituting Eq. (17) into Eq. (16), and introducing the appropriate relation between the solution values, gives

$$C_0 U_m = C_1 U_{m-1} + \bar{h}(G_0 F_m + G_1 F_{m-1}) \quad (18)$$

where,

$$C_0 = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{8(\rho+6)}{\rho+25} & 1 & 0 \\ -\frac{20(\rho-15)}{3\rho+137} & -\frac{30(\rho+10)}{3\rho+137} & 1 \end{pmatrix}, \; C_1 = \begin{pmatrix} -\frac{2(\rho-1)}{\rho+11} & -\frac{3(\rho+3)}{\rho+11} & \frac{6(\rho+3)}{\rho+11} \\ \frac{(\rho-3)}{\rho+25} & -\frac{8(\rho-2)}{\rho+25} & -\frac{36}{\rho+25} \\ -\frac{2(\rho-6)}{3\rho+137} & -\frac{15(\rho-5)}{3\rho+137} & -\frac{20(\rho-10)}{3\rho+137} \end{pmatrix}$$

$$G_0 = \begin{pmatrix} 0 & -\frac{6\rho}{\rho+11} & 0 \\ 0 & 0 & -\frac{12\rho}{\rho+25} \\ 0 & 0 & 0 \end{pmatrix}, \; G_1 = \begin{pmatrix} \frac{6}{\rho+11} & 0 & 0 \\ 0 & \frac{12}{\rho+25} & 0 \\ -\frac{60\rho}{3\rho+137} & 0 & \frac{60}{3\rho+137} \end{pmatrix}$$

The stability polynomial of the method can then be obtained from

$$Det\left[\left(C_0 - \bar{h}G_1\right) - \left(C_1 + \bar{h}G_0\right)\right] \quad (19)$$

which is equivalent to

$$\left[\left[\begin{pmatrix} 1 & 0 & 0 \\ -\frac{8(\rho+6)}{\rho+25} & 1 & 0 \\ -\frac{20(\rho-15)}{3\rho+137} & -\frac{30(\rho+10)}{3\rho+137} & 1 \end{pmatrix} - \bar{h}\begin{pmatrix} \frac{6}{\rho+11} & 0 & 0 \\ 0 & \frac{12}{\rho+25} & 0 \\ -\frac{60\rho}{3\rho+137} & 0 & \frac{60}{3\rho+137} \end{pmatrix}\right] t - \left[\begin{pmatrix} -\frac{2(\rho-1)}{\rho+11} & -\frac{3(\rho+3)}{\rho+11} & \frac{6(\rho+3)}{\rho+11} \\ \frac{(\rho-3)}{\rho+25} & -\frac{8(\rho-2)}{\rho+25} & -\frac{36}{\rho+25} \\ -\frac{2(\rho-6)}{3\rho+137} & \frac{15(\rho-5)}{3\rho+137} & -\frac{20(3\rho-10)}{3\rho+137} \end{pmatrix} + \bar{h}\begin{pmatrix} 0 & -\frac{6\rho}{\rho+11} & 0 \\ 0 & 0 & -\frac{12\rho}{\rho+25} \\ 0 & 0 & 0 \end{pmatrix}\right]\right] = 0 \quad (20)$$

After evaluating Eq. (20), the resulting stability polynomial is obtained as

$$R(t,\bar{h}) = \left(\frac{1}{(\rho+11)(\rho+25)(3\rho+137)}\right)(4320h^3\rho^3 t - 4320h^3 t^3 - 3600h^2\rho^3 t + 720h^2\rho^2 t^2 + 1296h^2\rho t^3 - 144h\rho^3 t^2$$

$$+1296\bar{h}^2\rho t^3 - 144\bar{h}\rho^3 t^2 - 114\bar{h}\rho^2 t^3 + 3\rho^3 t^3 + 144\bar{h}^2\rho^3 + 3600\bar{h}^2\rho^2 t + 34560\bar{h}^2\rho t^2$$
$$+26784\bar{h}^2 t^3 - 198\bar{h}\rho^3 t - 6816\bar{h}\rho^2 t^2 - 5472\bar{h}\rho t^3 - 1398\rho^3 t^2 + 245\rho^2 t^3 - 864\bar{h}^2\rho^2$$
$$-6480\bar{h}^2\rho t - 21600\bar{h}^2 t^2 - 792\bar{h}\rho^3 - 16572\bar{h}\rho^2 t - 118800\bar{h}\rho t^2 - 55134\bar{h}t^3 + 249\rho^3 t$$
$$-20214\rho^2 t^2 + 5757\rho t^3 + 4224\bar{h}\rho^2 + 3798\bar{h}\rho t - 69120\bar{h}t^2 + 1146\rho^3 + 25383\rho^2 t$$
$$-35490\rho t^2 + 37675t^3 - 1512\bar{h}\rho - 5508\bar{h}t - 5414\rho^2 + 27303\rho t - 32178t^2 + 2430\rho$$
$$-5415t - 82) = 0 \qquad (21)$$

Zero Stability Analysis

For zero-stability analysis, we set $\rho = -\frac{1}{2}$ in Eq. (21). This gives

$$R(t,\bar{h}) = t^3 - \frac{19972}{13279}t^3\bar{h} + \frac{69696}{92953}t^3\bar{h}^2 - \frac{11520}{92953}t^3\bar{h}^3 - \frac{51498}{92953}t^2 - \frac{221568}{92953}t^2\bar{h} - \frac{6240}{92953}t^2\bar{h}^2$$
$$-\frac{34005}{92953}t - \frac{59106}{92953}t\bar{h} + \frac{17520}{92953}t\bar{h}^2 - \frac{1440}{92953}t\bar{h}^3 - \frac{624}{92953}\bar{h}^2 - \frac{104}{1897}\bar{h} - \frac{7450}{92953} = 0 \qquad (22)$$

The first characteristic polynomial is obtained by setting $\bar{h} = 0$ in the resulting stability polynomial.

Hence,

$$t^3 - \frac{51498}{92953}t^2 - \frac{34005}{92953}t - \frac{7450}{92953} = 0 \qquad (23)$$

Solving the cubic characteristic equation gives the roots

$$t = 1,\ -0.2229890375 - 0.17442454231i,\ 0.2229890375 + 0.17442454231i$$

with corresponding moduli

$$t = 1, 0.2831042772638 \text{ and } 0.2831042772638.$$

Since all the roots satisfy $|\xi| \le 1$, and each root lying on the unit circle is simple, the root condition is satisfied. Therefore, according to Definition 4, the DIE3SBBDF method is zero-stable.

Consistency Analysis

From Definition 6, the DIE3SBBDF method is consistent if its order is at least one. Since the order of the method has already been established as five in section 3, the method satisfies the required order condition.

Alternatively, consistency can be verified directly from the order conditions. In particular, the conditions

$$\sum_{j=0}^{5} A_j = 0 \text{ and } \sum_{j=0}^{5} jA_j = \sum_{j=0}^{5} D_j \qquad (25)$$

where A_j and D_j are constant (column) matrices, must be satisfied. .

Let $A_0, A_1, \ldots, A_5$, the coefficient matrices be as defined previously. From the formulation in Eq. (21), we obtain

$$\sum_{j=0}^{5} A_j = A_0 + A_1 + A_2 + A_3 + A_4 + A_5$$

$$= \begin{pmatrix} \frac{2(\rho-1)}{\rho+11} \\ -\frac{(\rho-3)}{\rho+25} \\ \frac{2(\rho-6)}{3\rho+137} \end{pmatrix} + \begin{pmatrix} \frac{3(\rho+3)}{\rho+11} \\ \frac{8(\rho-2)}{\rho+25} \\ -\frac{15(\rho-5)}{3\rho+137} \end{pmatrix} + \begin{pmatrix} -\frac{6(\rho+3)}{\rho+11} \\ \frac{36}{\rho+25} \\ -\frac{20(3\rho-10)}{3\rho+137} \end{pmatrix} + \begin{pmatrix} 1 \\ -\frac{8(\rho-2)}{\rho+25} \\ -\frac{20(\rho-15)}{3\rho+137} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ -\frac{30(\rho+10)}{3\rho+137} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0$$

Thus, the first consistency condition is satisfied.

Similarly, $\sum_{j=0}^{5} jA_j = 0 + 1A_1 + 2A_2 + 3A_3 + 4A_4 + 5A_5$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{3(\rho+3)}{\rho+11} \\ \frac{8(\rho-2)}{\rho+25} \\ -\frac{15(\rho-5)}{3\rho+137} \end{pmatrix} + 2\begin{pmatrix} -\frac{6(\rho+3)}{\rho+11} \\ \frac{36}{\rho+25} \\ -\frac{20(3\rho-10)}{3\rho+137} \end{pmatrix} + 3\begin{pmatrix} 1 \\ -\frac{8(\rho-2)}{\rho+25} \\ -\frac{20(\rho-15)}{3\rho+137} \end{pmatrix} + 4\begin{pmatrix} 0 \\ 1 \\ -\frac{30(\rho+10)}{3\rho+137} \end{pmatrix} + 5\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{6(\rho-1)}{\rho+11} \\ -\frac{12(\rho-1)}{\rho+25} \\ \frac{60(\rho-1)}{3\rho+137} \end{pmatrix}$$

$$\sum_{j=0}^{5} D_j = D_0 + D_1 + D_2 + D_3 + D_4 + D_5$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{6\rho}{\rho+11} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{12\rho}{\rho+25} \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{6}{\rho+11} \\ 0 \\ -\frac{60\rho}{\rho+137} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{12}{\rho+25} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{60}{\rho+137} \end{pmatrix} = \begin{pmatrix} -\frac{6(\rho-1)}{\rho+11} \\ -\frac{12(\rho-1)}{\rho+25} \\ -\frac{60(\rho-1)}{3\rho+137} \end{pmatrix}$$

Therefore,

$$\sum_{j=0}^{5} jA_j = \sum_{j=0}^{5} D_j = \begin{pmatrix} -\frac{6(\rho-1)}{\rho+11} \\ -\frac{12(\rho-1)}{\rho+25} \\ -\frac{60(\rho-1)}{3\rho+137} \end{pmatrix}.$$

The second consistency condition is also satisfied.

Hence, the DIE3SBBDF method is consistent.

Since the DIE3SBBDF method is both zero-stable and consistent, it satisfies the necessary and sufficient conditions for convergence of a linear multistep method. Therefore, by the fundamental convergence theorem for

linear multistep methods (Lambert, 1973), the DIE3SBBDF method is convergent.

CONCLUSION

The order and convergence properties of the diagonally implicit extended 3-point superclass block backward differentiation formula (DIE3SBBDF) developed by Yusuf et al. (2026) for solving first-order stiff initial value problems have been investigated. The order analysis shows that the method is of fifth order. Furthermore, the consistency and zero-stability properties of the method have been established.

Since the DIE3SBBDF method satisfies both consistency and zero-stability, it follows from the fundamental convergence theorem for linear multistep methods that the method is convergent. These results provide the theoretical justification for the use of the DIE3SBBDF method in the numerical solution of first-order stiff initial value problems.

REFERENCES

Abdullahi, M., & Musa, H. (2021). Order and convergence of the enhanced 3-point fully implicit superclass of block backward differentiation formula for solving first-order stiff initial value problems. *FUDMA Journal of Sciences*, **5**(2), 579–584.

Alhassan, B., Musa, H., Yusuf, H., Adamu, A., Bello, A., & Hamisu, A. (2024). Derivation and implementation of A-stable diagonally implicit hybrid block method for the numerical integration of stiff ordinary differential equations. *Matrix Science Mathematic*, **8**(2), 24–32.

Alhassan, B., Yusuf, H., & Musa, H. (2024). Analysis of convergence and stability properties of diagonally implicit 3-point block backward differentiation formula for first-order stiff initial value problems. *UMYU Scientifica*, **3**(2), 186–202. https://doi.org/10.56919/usci.2432.021

Alhassan, B., Yusuf, H., Musa, H., & Abasi, N. (2023). A new fifth-order variable step-size block backward differentiation formula with off-step points for the numerical solution of stiff ordinary differential equations. *Applied Mathematics and Computational Intelligence*, **12**, 94–121.

Cash, J. R. (1980). On the integration of stiff systems of ODEs using extended backward differentiation formula. *Numerische Mathematik*, **34**, 235–246.

Cash, J. R. (1999). Modified extended backward differentiation formulae for the numerical solution of stiff initial value problems in ODEs and DEs. *Journal of Computational and Applied Mathematics*, 125, 117–130.

Curtiss, C. F., & Hirschfelder, J. O. (1952). Integration of stiff equations. Proceedings of the National Academy of Sciences of the United States of America, **38**, 235–243.

Lambert, J.D., (1973). Computational Methods in Ordinary Differential Equations. Wiley, New York.

Musa, H. (2013). The convergence and order of the 2-point improved block backward differentiation formula. *IOSR Journal of Mathematics*, **7**, 61–67.

Musa, H., & Bala, N. (2019). 3-point diagonally implicit superclass of block backward differentiation formula for solving stiff initial value problems. *Dutse Journal of Pure and Applied Sciences*, (1b), 1–10.

Musa, H., Bature, B., & Ibrahim, L. K. (2016a). Convergence of diagonally implicit superclass of block backward differentiation formula. Proceedings of the 53rd Annual Conference of the Mathematical Association of Nigeria, Ahmadu Bello University, Zaria, Nigeria.

Musa, H., Bature, B., & Ibrahim, L. K. (2016b). Derivation of diagonally implicit superclass of block backward differentiation formula for solving stiff initial value problems. Book of Abstracts, 35th Annual Conference of the Nigerian Mathematical Society.

Musa, H., Suleiman, M. B., & Ismail, F. (2015). An implicit 2-point block extended backward differentiation formulas for solving stiff IVPs. *Malaysian Journal of Mathematical Sciences*, **9**, 33–51.

Musa, H., Suleiman, M. B., & Senu, N. (2011). A-stable 2-points block extended backward differentiation formulae for solving ODEs. AIP Conference Proceedings, 1450, 254–258.

Musa, H., Suleiman, M. B., & Senu, N. (2012). Fully implicit 3-point block extended backward differentiation formulae for solving ODEs. *Applied Mathematical Sciences*, **6**(85), 4211–4228.

Musa, H., Suleiman, M. B., Zainal, S. Z., Ismail, F., Senu, N., & Ibrahim, Z. B. (2012). The convergence and order of the 3-point block extended backward differentiation formula. *ARPN Journal of Engineering and Applied Sciences*, **7**, 1539–1545.

Musa, H., & Unwala, M. A. (2019). Extended 3-point superclass of block backward differentiation formula for

solving stiff IVPs. *Abacus (Mathematics Science Series)*, **44**(1), 1–12.

Sagir, A. M., Abdullahi, M., Suleiman, S., & Sule, B. (2022). An A-stable block integrator scheme for the solution of first-order system of IVPs of ordinary differential equations. *Asian Journal of Probability and Statistics*, **16**(4), 11–28.

Suleiman, M. B., Musa, H., Ismail, F., Senu, N., & Ibrahim, Z. B. (2014). A new superclass of block backward differentiation formulas for stiff ODEs. *Asian-European Journal of Mathematics*, **7**, 1350034.

Yusuf, H., Bello, A., & Yahaya, U. (2025). Order and convergence analysis of the new fixed-coefficient diagonally implicit block backward differentiation formula for numerical solution of stiff ordinary differential equations. *Journal of Basic and Applied Scientific Research (JOBASR)*, **1**(1), 138–145. https://doi.org/10.4314/jobasr.v1i1.15s.

Yusuf, H., Musa, H., & Alhassan, B. (2026). Diagonally implicit extended 3-point superclass block backward differentiation formula for solving stiff initial value problems. *Matrix Science Mathematic*, **10**(1), 42–49.