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A Hybrid Transmuted Alpha-G Lindley Family of Distribution: Properties, and Applications

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ABSTRACT
The study of probability distributions is essential for modelling and predicting real-world phenomena across diverse fields such as medicine, engineering, and computer science. However, classical distributions often struggle to represent complex data characteristics, including skewness, kurtosis, and heavy-tailed behavior. To address these limitations, this study introduces a novel hybrid family of distributions by combining two generators, the Transmutation-G and Alpha Power-G families, resulting in a more flexible and adaptable statistical framework. The proposed hybrid transmuted alpha-G Lindley (HTA-G Lindley) family leverages the strengths of both approaches, and its statistical properties are thoroughly derived. The maximum likelihood estimation method is also presented for parameter estimation within the proposed family. The effectiveness of the new models is demonstrated through numerical analyses using a real-life dataset and the results of the sub-models, indicate that the TAEWEL model provides a superior fit compared with other competing model, highlighting its potential for modelling complex datasets.

KEYWORDS:
Transmuted Alpha-G family,
Moments generating function,
Quantile function and
Maximum likelihood estimation.

INTRODUCTION

The study of probability distributions is essential for modelling and predicting real-world phenomena in many fields, such as responses to treatments in medicine, failure time analysis in engineering, duration analysis in Economics, and climate data pattern in environmental science (Odeyale *et al.* 2023). However, classical distributions often have limited flexibility in describing complex datasets characterized by skewness, heavy tails, and varying hazard-rate behaviours. For instance, medical datasets related to Anemia, disease outbreak, and growth hormone treatments frequently display skewed distributions, where the data may show a monotonically increasing or decreasing hazard pattern. Classical distributions, while foundational in statistics, often fail to capture the complexity of these datasets (Bashiru *et al.* 2025). The Lindley distribution is useful for positively skewed lifetime data but is restricted by its single parameter and limited ability to accommodate diverse data patterns.

Several notable families of distribution have been developed in recent years, such as the Lindley Family of Distributions by Cakmakyapan and Ozel (2016). The Odd Lindley-G Family of Distributions by Gomes *et al.* (2017),), the generalized odd gamma-G family presented by Hosseini et al. (2018), the generalized Kumaraswamy generalized (GK-G) family was presented by Nofal *et al.* (2019), Alpha Power Transformed Weibull-G Family of Distributions by Elbata *et al.* (2020), 'the Transmuted Topp Leone Flexible Weibull (TTLFW) distribution with applications to reliability and lifetime data' by Alobaidi *et al.* (2021), the weighted Weibull-G family of distributions by Hassan et al. (2022), 'Topp-Leone Cauchy Family of Distributions with Applications in Industrial Engineering' by Atchadé, *et al.* (2023), the development of exponentiated cosine Topp-Leone generalized family of distributions and its applications to lifetime data by Osi *et al.* (2024), a new family of Lindley distributions by Opone *et al.* (2025), the transmuted Rayleigh-X Family of Distributions by Abdullahi *et al.*, (2025), a new family of Lindley distributions (Festus *et al.* 2025), a hybrid distribution of Weibull, Exponential and Gamma distribution by Izomo

et al. (2025), and a Hybrid Cosine Inverse Lomax-G Family of Distributions by Bashiru *et al.* (2025).

Despite these advances, the demand for diverse distribution families continues because no single model can universally fit all data types. To overcome these limitations, generalized and transformed distribution families have been developed to provide greater modelling flexibility. The Alpha-Power transformation introduces an additional shape parameter that can modify the characteristics of a baseline distribution. Similarly, transmutation techniques enhance distributional flexibility by altering skewness, kurtosis, and tail behaviour. Combining these transformation mechanisms provides an opportunity to exploit their complementary advantages. In response to this need, this study aims to introduce a novel Hybrid Transmuted Alpha-G Lindley (HTA-G Lindley) family of distributions. The proposed family combines the Alpha-Power-G generator with the transmuted power generator and the Lindley baseline distribution.

Baseline Lindley Distribution

The PDF can be given as:

$$g(x;\theta)=\frac{\theta^2}{1+\theta}(1+x)e^{-\theta x},\quad x>0,\theta>0, \qquad (1)$$

with CDF

$$G(x;\theta)=1-\left(1+\frac{\theta x}{1+\theta}\right)e^{-\theta x}, \qquad (2)$$

Where θ is a scale parameter

Alpha-Power Transformation (APT)

According to Mahdavi and Kundu (2017), the CDF of the APT distribution is defined as:

$$F(x;\alpha)=G(x)^{\alpha},\alpha>0,\alpha\neq 1. \qquad (3)$$

$$f(x;\alpha)=\alpha g(x)G(x)^{\alpha-1}. \qquad (4)$$

Where α is a shape parameter

The Quadratic Rank Transmutation

According to Shaw and Buckley (2007), a random variable X is said to have a transmuted distribution function if its pdf and cdf are respectively given by;

METHODOLOGY

The proposed Transmuted Alpha-G Lindley family of distributions is derived by combining the alpha Power Transformed Weibull-G Family of Distributions by Elbata *et al. (*2020), with the transmuted-X family of distributions (Abdullahi *et al.* 2025) and a new family of Lindley distributions (Festus *et al.* 2025). The methodology involves the following steps:

$$F(x)=(1+\lambda)G(x)^{\alpha}-\lambda G(x)^{2\alpha} \qquad (5)$$

$$f(x)=\alpha g(x)G(x)^{\alpha-1}\left[1+\lambda-2\lambda G(x)^{\alpha}\right] \qquad (6)$$

where $\lambda\in(-1,1)$ is the transmutation parameter.

Hence, the proposed Hybrid Transmuted Alpha-G Lindley (HTA Lindley) family of distribution is define by the combined transformation given as;

$$W(G)=-\log(G) \qquad (7)$$

and the hybrid generator of the CDF is define as:

$$R_{\alpha,\lambda}(t)=(1+\lambda)e^{-\alpha t}-\lambda e^{-2\alpha t}. \qquad (8)$$

Therefore,

$$r_{\alpha,\lambda}(t)=\alpha e^{-\alpha t}\left[1+\lambda-2\lambda e^{-\alpha t}\right]. \qquad (9)$$

$$F(x)=\int_{a}^{W[G(x)]} r_{\alpha,\lambda}(t)dt. \qquad (10)$$

Since

$$W(G(x))=-\log G(x), \qquad (11)$$

Expand the integral:

$$F(x)=(1+\lambda)\left[1-e^{-\alpha[-\log G(x)]}\right]+\lambda\left[e^{-2\alpha[-\log G(x)]}-1\right]. \qquad (12)$$

Hence, the CDF of HTA-G Lindley family of distribution is given as:

$$F(x)=(1+\lambda)\left[1-\left(1+\frac{\theta x}{1+\theta}\right)e^{-\theta x}\right]^{\alpha}-\lambda\left[1-\left(1+\frac{\theta x}{1+\theta}\right)e^{-\theta x}\right]^{2\alpha}. \qquad (13)$$

Its corresponding pdf is given as:

$$f(x)=\frac{\alpha\theta^2}{1+\theta}(1+x)e^{-\theta x}\left[1-\left(1+\frac{\theta x}{1+\theta}\right)e^{-\theta x}\right]^{\alpha-1}\left\{1+\lambda-2\lambda\left[1-\left(1+\frac{\theta x}{1+\theta}\right)e^{-\theta x}\right]^{\alpha}\right\}. \qquad (14)$$

Checking the Validity of the *HTA-G Lindley* Family of Distributions

It is important to check whether the proposed pdf of the new family is valid or not.

Proposition 1: The pdf of the HTA -G Lindley family of distribution is well-defined as:

using (3.14), it suffices to show that;

$$\int_0^{\infty} f\left(x;\alpha,\theta,\lambda\right)dx = 1 \qquad (15)$$

$$\int_0^{\infty} f(x;\alpha,\theta,\lambda)dx = \int_0^{\infty} \alpha g\left(x\right)G\left(x\right)^{\alpha-1}\left[1+\lambda-2\lambda G\left(x\right)^{\alpha}\right]dx. \quad (16)$$

Let $u = G\left(x\right),\ du = g\left(x\right)dx,$ As $u \to 0, G\left(x\right) \to 0;$ As $u \to \infty, G\left(\infty\right) \to 1.$

Hence,

$$\int_0^1 \alpha u^{\alpha-1}\left(1+\lambda-2\lambda u^{\alpha}\right)du.$$

$$= \alpha\left(1+\lambda\right)\int_0^1 u^{\alpha-1}du - 2\alpha\lambda\int_0^1 u^{2\alpha-1}du. \quad (17)$$

Substitute the pdf of (3.14) as;

$$g\left(x\right) = \frac{\theta^2}{1+\theta}\left(1+x\right)e^{-\theta x},$$ and

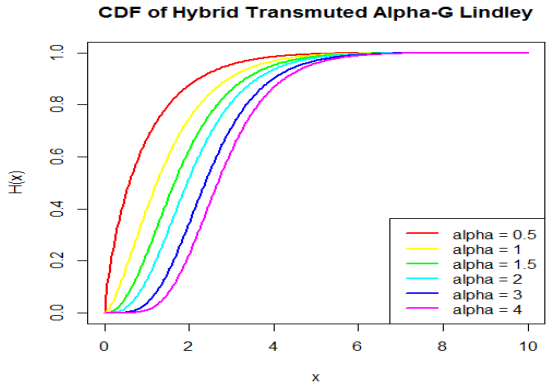
$$G\left(x\right) = \left[1-\left(1+\frac{\theta x}{1+\theta}\right)e^{-\theta x}\right]$$

Then, we obtain:

$$\int_0^{\infty} f\left(x\right)dx = \alpha\left(1+\lambda\right)\frac{1}{\alpha} - 2\alpha\lambda\left.\frac{1}{2\alpha}\right|_0^1 \quad (18)$$

$$= \left(1+\lambda\right)-\lambda = 1.$$

Thus, the integral evaluates to 1, confirming that the PDF is valid.



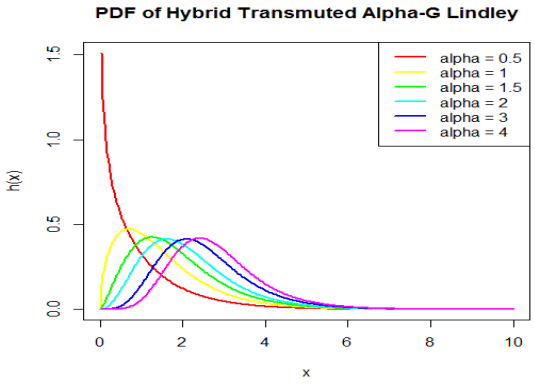


Fig. 1: The CDF and PDF of (HTA-G Lindley) family of *distribution*

Survival Function

Survival function is the probability that a system or an individual will survive beyond a given time. Mathematically, the survival function of the *HTA-G* Lindley family of distribution is given as:

$$S\left(x;\alpha,\lambda\right) = 1-\left[\left(1+\lambda\right)G\left(x\right)^{\alpha} - \lambda G\left(x\right)^{2\alpha}\right], 19)$$

Hazard Rate Function

The hazard rate function of the *HTA-G Lindley* family of distribution is given as:

$$h\left(x;\alpha,\theta,\lambda\right) = \frac{\alpha g\left(x\right)G\left(x\right)^{\alpha-1}\left[1+\lambda-2\lambda G\left(x\right)^{\alpha}\right]}{1-\left[\left(1+\lambda\right)G\left(x\right)^{\alpha} - \lambda G\left(x\right)^{2\alpha}\right]} \quad (20)$$

Expansion of the CDF and PDF for the HTA-G Lindley Family of distribution

We describe the CDF of the generating function for the HTA-G Lindley family of distributions. Upon evaluating the CDF using Equation (3.20), consider the representation of the function as a series.

$$F(x)=(1+\lambda)\left[1-\left(1+\frac{\theta x}{1+\theta}\right)e^{-\theta x}\right]^{\alpha}-\lambda\left[1-\left(1+\frac{\theta x}{1+\theta}\right)e^{-\theta x}\right]^{2\alpha}.$$

Let, $P=\left[1-\left(1+\frac{\theta x}{1+\theta}\right)e^{-\theta x}\right]^{\alpha}$ and

$$q=\left[1-\left(1+\frac{\theta x}{1+\theta}\right)e^{-\theta x}\right]^{2\alpha}$$

Using Maclaurin's series expansion for *p* and *q*,

$$p=\sum_{i=0}^{\infty}\frac{(-1)^{i}\left[1-\left(1+\frac{\theta x}{1+\theta}\right)e^{-\theta x}\right]^{\alpha i}}{i!},$$

$$q=\sum_{i=0}^{\infty}\frac{(-1)^{i}\left[1-\left(1+\frac{\theta x}{1+\theta}\right)e^{-\theta x}\right]^{2\alpha i}}{i!} \quad (21)$$

Substituting the values of *p* and *q* in the *CDF* yields;

$$F(x)=(1+\lambda)\sum_{i=0}^{\infty}\frac{(-1)^{i}\left[1-\left(1+\frac{\theta x}{1+\theta}\right)e^{-\theta x}\right]^{\alpha i}}{i!}-\lambda\sum_{i=0}^{\infty}\frac{(-1)^{i}\left[1-\left(1+\frac{\theta x}{1+\theta}\right)e^{-\theta x}\right]^{2\alpha i}}{i!} \quad (22)$$

$$=\sum_{i,j=0}^{\infty}\frac{(-1)^{\alpha(i+j)}}{i!}\binom{-\alpha i}{j}(1+\lambda)G(x)^{\alpha(i+j)}-\lambda G(x)^{2\alpha(i+j)}$$

Hence,

$$F(x)=\sum_{i,j=0}^{\infty}m_{i,j}G(x)^{\alpha}=\sum_{i,j=0}^{\infty}m_{i,j}\prod_{\alpha}(x). \quad (24)$$

is the *CDF* of the *(HTA-G Lindley)* family of distribution with shape parameter α.

Statistical Properties of HTA-G Lindley Family of Distributions

When new families of distributions are proposed it is frequently wonted to derive some of the statistical properties of the new family, such as: moments, moment-generating function, incomplete moment, quantile function, and order statistics

Moments

The r^{th} ordinary moment of the *(HTA-G Lindley)* family is define by the following;

$$E(X^{r})=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}m_{i,j}E(X_{i,j}^{r}) \quad (25)$$

$$\mu_{r}^{'}=E(x^{r})=\int_{0}^{\infty}x^{r}\sum_{i=0}^{\infty}m_{i,j}\alpha g(x)G(x)^{\alpha-1}\left[1+\lambda-2\lambda G(x)^{\alpha}\right]dx \quad (26)$$

$$=\int_{0}^{\infty}x^{r}\sum_{i=0}^{\infty}m_{i,j}\prod_{\alpha}(x)dx \quad (27)$$

Where, $\mu^{r}=E(X_{i,j}^{r})=\sum_{i,j=0}^{\infty}m_{i,j}E(X_{i,j}^{r})$ (28)

Moment Generating Function (MGF)

The moment generating function of the *HTA-G Lindley* family of distribution is define by the following;

$$M_X(t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}m_{i,j}E\left(e^{tx_{i,j}}\right) \quad (29)$$

then,

$$M_X(t)=\int_0^{\infty}e^{tx_{i,j}}\sum_{i=0}^{\infty}m_{i,j}\alpha g(x)G(x)^{\alpha-1}\left[1+\lambda-2\lambda G(x)^{\alpha}\right]dx \quad (30)$$

$$=\sum_{i=0}^{\infty}m_{i,j}\int_0^{\infty}e^{tx_{i,j}}\prod\nolimits_{\alpha}(x)dx \quad (31)$$

Hence, the MGF is written as;

$$M_X(t)=\sum_{i,j}^{\infty}m_{i,j}E\left(e^{tx_{i,j}}\right) \quad (32)$$

Incomplete moment

The r^{th} Incomplete moment of the *HTA-G Lindley* family of *distribution* is define by the following;

$$I_r(t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}m_{i,j}I_{i,j}(t),$$

$$I_r(t)=\int_0^{\infty}x^r h(x;\alpha,\theta,\lambda)dx \quad (33)$$

$$I_r(t)=\int_0^{\infty}x^r\sum_{i=0}^{\infty}m_{i,j}\alpha g(x)G(x)^{\alpha-1}\left[1+\lambda-2\lambda G(x)^{\alpha}\right]dx$$

$$I_r(t)=\sum_{i,j=0}^{\infty}m_{i,j}\int_0^{\infty}x^r\prod\nolimits_{\alpha}(x)dx=\sum_{i,j=0}^{\infty}m_{i,j}I_{i,j}(t) \quad (34)$$

Quantile Function of the *HTA-G Lindley* Family of Distribution

The quantile function of the *HTA-G Lindley* say $Q(u)=F^{-1}(u)$ for $u\in(0,1)$, given that $\lambda\neq 0$ and $\alpha\neq 0$ are the solution of the non-linear equation.

$$u=(1+\lambda)\left[1-\left(1+\frac{\theta x}{1+\theta}\right)e^{-\theta x}\right]^{\alpha}-\lambda\left[1-\left(1+\frac{\theta x}{1+\theta}\right)e^{-\theta x}\right]^{2\alpha},\ 0<u<1. \quad (35)$$

Let $t=\left[1-\left(1+\frac{\theta x}{1+\theta}\right)e^{-\theta x}\right]^{\alpha}$,

Using quadratic formula and substitute t into (3.52), we have:

$$u=(1+\lambda)t-\lambda t^2 \quad (36)$$

Rearrange to quadratic form:

$$\lambda t^2-(1+\lambda)t+u=0 \quad (37)$$

Solve:

$$t=\left[(1+\lambda)-\sqrt{\left((1+\lambda)^2-4\lambda u\right)}\right]\Big/(2\lambda) \quad (38)$$

Then the quantile is:

$$Q(u)=G^{-1}\left(t^{1/\alpha}\right) \quad (39)$$

This requires the inverse baseline *CDF* G^{-1}.

where $G^{-1}(.)$ represent the quantile function of the baseline distribution with random variable x.

The median of the *HTA Lindley-G* can be obtained by setting $u=0.5$ in (3.39) as:

$$Q_x(0.5)=G^{-1}\left(t^{1/\alpha}\right), \quad (40)$$

Distribution of Order Statistic

Let $X_1, X_2, ..., X_n$ be n independent random variable from the *HTA-G Lindley* family of distribution and let $X_{(1)} \le X_{(2)} \le ... \le X_{(n)}$ be their corresponding order statistic. The pdf of $X_{r:n}$ is given as:

$$f_{r:n}(x) = \frac{n!}{(p-1)!(n-p)!} f(x;\alpha,\theta,\lambda) F(x;\alpha,\theta,\lambda)^{p-1} \left[1 - F(x;\alpha,\theta,\lambda)\right]^{n-p} \quad (41)$$

Then, the order statistics of the *HTA* Lindley-*G* distribution is given by:

$$f_{r:n}(x) = \sum_{k=0}^{n-p} \frac{(-1)^k n!}{(p-1)!(n-p-k)!k!} \left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} m_{i,j} \prod_{\alpha}(x)\right) \left(1 + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} m_{i,j} \prod_{\alpha}(x)\right)^{p+k-1} \quad (42)$$

Parameter Estimation

In this section, the method of maximum likelihood estimation *(MLEs)* will be derive for estimation of parameters of the *HTA-G Lindley* family of distribution. Let $X_1, X_2, ..., X_n$ be an independent and identically distributed observed random sample of size n from the *HTA-G Lindley* family of distributions with parameters α, θ and λ. Then, the log-likelihood function based on observed sample for the vector of parameter $\Theta = (\alpha, \theta, \lambda)^T$ is given by

$$L(\alpha,\theta,\lambda) = \prod_{i=1}^{n} f(x_i). \quad (43)$$

Then, taking the log-likelihood α, θ and λ is given as:

$$\ell(\alpha,\theta,\lambda) = n\log\alpha + 2n\log\theta - n\log(1-\theta) + \sum_{i=1}^{n}\log(1+x_i) - \theta\sum_{i=1}^{n} x_i + (\alpha-1)\sum_{i=1}^{n}\log G_i + \sum_{i=1}^{n}\log\left(1+\lambda-2\lambda G_i^{\alpha}\right). \quad (44)$$

The components of the score vector, $\nabla\ell = \left(\frac{\partial\ell}{\partial\alpha}, \frac{\partial\ell}{\partial\theta}, \frac{\partial\ell}{\partial\lambda}\right)$ can be obtained differentiating ℓ partially with respect to the parameters α, θ and λ equating the result to zero.

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\hat{\alpha}} + \sum_{i=0}^{n}\log G_i - 2\lambda\sum_{i=0}^{n}\frac{G_i^{\hat{\alpha}}\log G_i}{1+\lambda-2\lambda G_i^{\hat{\alpha}}} = 0. \quad (45)$$

$$\frac{\partial\ell}{\partial\lambda} = \sum_{i=1}^{n}\frac{1-2G_i^{\hat{\alpha}}}{1+\hat{\lambda}-2\hat{\lambda}G_i^{\hat{\alpha}}} = 0 \quad (46)$$

$$\frac{\partial\ell}{\partial\theta} = \frac{2n}{\hat{\theta}} - \frac{n}{1+\hat{\theta}} - \sum_{i=1}^{n} x_i + (\hat{\alpha}-1)\sum_{i=1}^{n} D_i - 2\hat{\lambda}\hat{\alpha}\sum_{i=1}^{n}\frac{G_i^{\hat{\alpha}}}{1+\hat{\lambda}-2\hat{\lambda}G_i^{\hat{\alpha}}} = 0 \quad (47)$$

The MLEs, say $\hat{\Theta} = \left(\hat{\alpha}, \hat{\theta}, \hat{\lambda}\right)$ of $\Theta = (\alpha, \theta, \lambda)^T$ can be obtained by solving the system of nonlinear equations (45) and (47). These equations cannot be solved analytically and it needed iterative techniques such as Newton-Raphson algorithm.

Some new distribution arising from the HTA-G family of distributions

This section presents four special models of the HTA-G Lindley family, which generalize several well-known distributions. These models are derived using the Exponentiated Exponential (EE), Exponentiated Inverse Exponential (EIE), Exponentiated Lomax Exponential (ELE), and Exponentiated Weibull Exponential (EWE), distributions as baseline distributions.

The Hybrid Transmuted Alpha Exponentiated Exponential Lindley (HTA-EEL) Distribution

The exponentiated exponential (EEx) distribution has cdf and pdf given as

$$G_{EEx}(x) = \left(1 - e^{-\beta x}\right)^{\sigma}, \quad (48)$$

$$g(x)=\beta\sigma\left(1-e^{-\beta x}\right)^{\sigma-1}.$$
$$x>0,\beta>0,\sigma>0. \quad (49)$$

Then, the (HTA-EExL) Distribution is obtained as:

$$F(x)=\left\{1-\left[1+\frac{\theta\left(G_{EEx}(x)\right)}{1+\theta}\right]e^{-\theta\left(G_{EEx}(x)\right)}\right\}^{\alpha}\left[1+\lambda-\lambda\left\{1-\left[1+\frac{\theta\left(G_{EEx}(x)\right)}{1+\theta}\right]e^{-\theta\left(G_{EEx}(x)\right)}\right\}^{\alpha}\right],\ x,\alpha,\beta,\sigma,\theta>0. \quad (50)$$

$$f(x)=\frac{\alpha\theta^2}{1+\theta}\left(g_{EEx}(x)\right)\left(G_{EEx}(x)\right)e^{-\theta\left(G_{EEx}(x)\right)}\left\{1-\left[1+\frac{\theta\left(G_{EEx}(x)\right)}{1+\theta}\right]e^{-\theta\left(G_{EEx}(x)\right)}\right\}^{\alpha-1}\left[1+\lambda-2\lambda\left\{1-\left[1+\frac{\theta\left(G_{EEx}(x)\right)}{1+\theta}\right]e^{-\theta\left(G_{EEx}(x)\right)}\right\}^{\alpha}\right] \quad (51)$$

The Hybrid Transmuted Alpha Exponentiated Inverse Exponential Lindley (HTA-EIExL) Distribution

The Exponentiated Inverse exponential (EIEx) distribution has cdf and pdf given as

$$G_{EIEx}(x;\beta,\theta)=\left(1-e^{-\left(\frac{\theta}{x}\right)}\right)^{\beta}\quad x>0,\beta,\theta>0 \quad (52)$$

$$g_{EIEx}(x;\beta,\theta)=\beta\left(\frac{\theta}{x^2}\right)\left(1-e^{-\left(\frac{\theta}{x}\right)}\right)^{\beta-1}$$
$$x>0,\beta,\theta>0 \quad (53)$$

Then, the (HTA-EIExL) Distribution is obtained as:

$$F(x)=\left\{1-\left[1+\frac{\theta\left(G_{EIEx}(x)\right)}{1+\theta}\right]e^{-\theta\left(G_{EIEx}(x)\right)}\right\}^{\alpha}\left[1+\lambda-\lambda\left\{1-\left[1+\frac{\theta\left(G_{EIEx}(x)\right)}{1+\theta}\right]e^{-\theta\left(G_{EIEx}(x)\right)}\right\}^{\alpha}\right], \quad (54)$$

$$f(x)=\frac{\alpha\theta^2}{1+\theta}\left(g_{EIEx}(x)\right)\left\{1-\left[1+\frac{\theta\left(G_{EIEx}(x)\right)}{1+\theta}\right]e^{-\theta\left(G_{EIEx}(x)\right)}\right\}^{\alpha-1}\left[1+\lambda-2\lambda\left\{1-\left[1+\frac{\theta\left(G_{EIEx}(x)\right)}{1+\theta}\right]e^{-\theta\left(G_{EIEx}(x)\right)}\right\}^{\alpha}\right] \quad (55)$$

The Hybrid Transmuted Alpha Exponentiated Lomax Exponential Lindley (HTA-ELExL) Distribution

The Exponentiated Lomax Exponential (ELEx) distribution has cdf and pdf given as

$$G_{ELEx}(x)=\left[1-(1+\theta x)^{-\beta}e^{-\eta x}\right]^{\rho} \quad (56)$$

$$g_{ELEx}(x)=\rho\left[\beta\theta+\eta(1+\theta x)\right](1+\theta x)^{-(\beta+1)}e^{-\eta x}\left[1-(1+\theta x)^{-\beta}e^{-\eta x}\right]^{\rho-1} \quad (57)$$

Then, the (HTA-ELExL) Distribution is obtained as:

$$F(x)=\left\{1-\left[1+\frac{\theta\left(G_{ELEx}(x)\right)}{1+\theta}\right]e^{-\theta\left(G_{ELEx}(x)\right)}\right\}^{\alpha}\left[1+\lambda-\lambda\left\{1-\left[1+\frac{\theta\left(G_{ELEx}(x)\right)}{1+\theta}\right]e^{-\theta\left(G_{ELEx}(x)\right)}\right\}^{\alpha}\right],\ x,\alpha,\beta,\theta,\lambda>0. \quad (58)$$

$$f(x)=\frac{\alpha\theta^2}{1+\theta}\left(g_{EIEx}(x)\right)\left\{1-\left[1+\frac{\theta\left(G_{ELEx}(x)\right)}{1+\theta}\right]e^{-\theta\left(G_{ELEx}(x)\right)}\right\}^{\alpha-1}\left[1+\lambda-2\lambda\left\{1-\left[1+\frac{\theta\left(G_{ELEx}(x)\right)}{1+\theta}\right]e^{-\theta\left(G_{ELEx}(x)\right)}\right\}^{\alpha}\right] \quad (59)$$

The Hybrid Transmuted Alpha Exponentiated Weibull Exponential Lindley (HTA-EWExL) Distribution

The Exponentiated Weibull Exponential (EWEx) distribution has cdf and pdf given as

$$G_{EWEx}(x)=\left[1-e^{-\gamma\left(e^{-\sigma x}-1\right)^{\beta}}\right]^{\eta} \quad (60)$$

$$g_{EWEx}(x)=\eta\gamma\beta\sigma e^{-\sigma x}\left(e^{-\sigma x}-1\right)^{\beta-1}e^{-\gamma\left(e^{-\sigma x}-1\right)^{\beta}}\left[1-e^{-\gamma\left(e^{-\sigma x}-1\right)^{\beta}}\right]^{\eta-1} \quad (61)$$

Then, the (HTA-EWExL) Distribution is obtained as:

$$F(x)=\left\{1-\left[1+\frac{\theta\left(G_{EWEx}(x)\right)}{1+\theta}\right]e^{-\theta\left(G_{EWEx}(x)\right)}\right\}^{\alpha}\left[1+\lambda-\lambda\left\{1-\left[1+\frac{\theta\left(G_{EWEx}(x)\right)}{1+\theta}\right]e^{-\theta\left(G_{EWEx}(x)\right)}\right\}^{\alpha}\right],\ x,\alpha,\beta,\theta,\lambda>0. \quad (62)$$

$$f(x)=\frac{\alpha\theta^{2}}{1+\theta}\left(g_{EWEx}(x)\right)\left\{1-\left[1+\frac{\theta\left(G_{EWEx}(x)\right)}{1+\theta}\right]e^{-\theta\left(G_{EWEx}(x)\right)}\right\}^{\alpha-1}\left[1+\lambda-2\lambda\left\{1-\left[1+\frac{\theta\left(G_{EWEx}(x)\right)}{1+\theta}\right]e^{-\theta\left(G_{EWEx}(x)\right)}\right\}^{\alpha}\right] \quad (63)$$

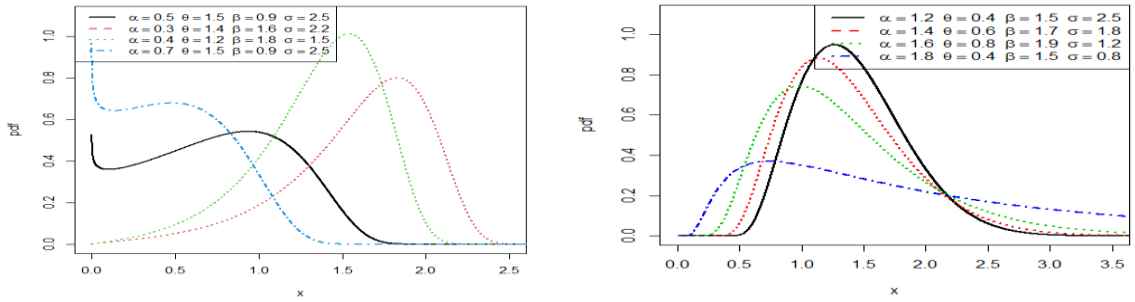


Figure 2: Density function of the TAEELD and TAEIELD for different values of the parameters

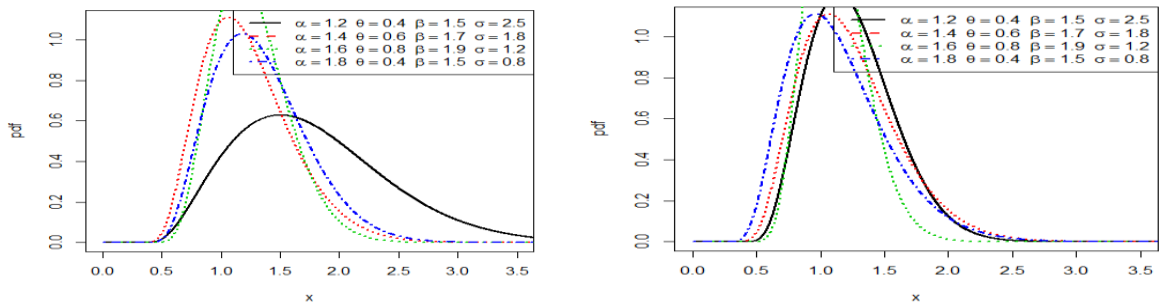


Figure 3: Density function of the TAELED and TAEWELD for different values of the parameters

Application of the HTA-G Lindley family of distributions data set

To evaluate the flexibility of the HTA-G Lindley family of distributions in fitting real world data from medical fields, we employ the Exponentiated Weibull Exponential Lindley Distribution belonging to the HTA-G Lindley family of distributions to fit the estimates of time from growth hormone treatment to target age in

children. This dataset was compared against the truncated exponentiated exponential exponentiated Weibull (TEEEW) distribution. The data was obtained from Oluyede et al. (2020). The dataset includes 35 observations. 2.15, 2.20, 2.55, 2.56, 2.63, 2.74, 2.81, 2.90, 3.05, 3.41, 3.43, 3.43, 3.84, 4.16, 4.18, 4.36, 4.42, 4.51,4.60, 4.61, 4.75, 5.03, 5.10, 5.44, 5.90, 5.96, 6.77, 7.82, 8.00, 8.16, 8.21, 8.72, 10.40, 13.20, 13.70.

We compared the Transmuted Alpha Exponentiated Weibull Exponential Lindley (TAEWEL) distribution, with the Truncated Exponentiated Exponential Exponentiated Weibull (TEEEW) distribution to assess the flexibility. For the purpose of model comparison, the fits of the distributions were compared based on the maximized log-likelihood (Log-Lik), Aikaike Information Criterion (AIC), Corrected Aikaike Information, Hannan-Quinn Information Criterion (HQIC), Cramér-von Mises (W), Anderson-Darling (A), and Kolmogorov-Smirnov (KS), and the model with the smaller information criterion is generally preferred. The (TAEWEL) shows is evidenced by its lowest AIC, CAIC, BIC, HQIC, W, A, and KS values, along with the highest associated p-value. The goodness-of-fit test results in Table 1 indicate that the TAEWEL model outperforms an existing TEEEW model in fitting the dataset. Collectively, these metrics confirm that the TAEWEL distribution provides the best fit for the data.

Table 1. The goodness-of-fit results for the Dataset

Model	MLEs	-2L	AIC	CAIC	BIC	HQIC	W	A	K-S	p-value
TAEWEL	$\hat{a}$: 3.8870	79.36	164.22	165.43	169.32	166.27	0.0735	0.1132	0.0978	0.9214
	$\hat{\beta}$: 2.5405									
	$\hat{\theta}$: 1.4991									
	$\hat{\gamma}$: 5.4408									
	$\hat{\eta}$: 0.4822									
	$\hat{\sigma}$: 0.4312									
	$\hat{\lambda}$: 0.3771									
TEEEW	$\hat{a}$: 0.6426	85.75	181.53	183.30	189.98	184.58	0.1604	1.0963	0.1158	0.6558
	$\hat{\beta}$: 1.3606									
	$\hat{\theta}$: 1.5128									
	$\hat{\gamma}$: 2.5314									
	$\hat{\eta}$: 0.1684									

DISCUSSION OF RESULT

A suitable model for analyzing lifetime data set can be investigated among several distributions by examining the model with the maximized log-likelihood value and the least value in terms of Information Criterion. Table 1 reveals that the Transmuted Alpha Exponentiated Weibull Exponential Lindley (TAEWEL) distribution, which belongs to the Hybrid Transmuted Alpha-G Lindley family of Distributions outperformed the (TEEEW) distribution in analyzing the data set and thus, can be employed as a better alternative to the (TEEEW) distribution in fitting real-life data sets.

CONCLUSION

A new class of Hybrid Transmuted Alpha-G Lindley family of Distributions with properties and application was introduced. Sub-families of the HTA-G Lindley family based on the quantile function of the Exponentiated exponential, Inverse exponential, Lomax exponential, and Weibull exponential distributions as well as some mathematical properties were derived. A growth hormone treatment data set was used to illustrate the applicability of the HTA-G Lindley of distributions and result obtained revealed that the TAEWEL distribution from the proposed HTA-G Lindley of distributions can be used as an alternative model to other existing distributions in modelling lifetime data sets.

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