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Dynamical Analysis of a Three-Species Model with Logistic Growth and Holling Type II Functional Responses in a Predator-Prey System

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ABSTRACT
This paper presents a modified three-species intraguild predator-prey model describing the interactions among lions (x), leopards (y), and hares (z) in an ecosystem. The linear model proposed by Kwaghkor *et al.* (2024) is modified by incorporating logistic growth terms for all populations and replacing the Type I functional responses with saturating Holling Type II functional responses. The resulting system of three nonlinear ordinary differential equations is analyzed for positivity, boundedness, existence of equilibria, and local asymptotic stability. Using the Jacobian matrix and the Routh–Hurwitz criterion, explicit stability conditions are derived for the trivial, single-species, and interior equilibria. Numerical simulations were performed with parameter sets adapted from the work of Kwaghkor et al. (2024). The modified model eliminates the unrealistic persistent oscillations observed in the work of Kwaghkor et al. (2024) linear system and demonstrates convergence to either a stable interior coexistence equilibrium depending on relative predation strengths and carrying capacities. The results confirm that the incorporation of density dependence and saturating functional responses produce qualitative realistic and mathematically robust dynamics. This enhanced model provides a more reliable framework for studying intraguild prey-predation in savanna ecosystems and offers practical insights for wildlife conservation.

Keywords:

Intraguild predation;
Equilibrium points;
Local stability;
Jacobian Matrix and
Routh-Hurwitz criterion.

INTRODUCTION

Predator-prey dynamics shape ecosystems, biodiversity, and population trends. The field began with Lotka (1925) and Volterra (1926), whose classic equations used exponential growth and linear (Type I) feeding, leading to unrealistic, never-ending oscillations.

To fix that, Rosenzweig and MacArthur (1963) introduced logistic prey growth and Holling's (1959) Type II saturating response creating the Rosenzweig-MacArthur model. Hastening and Powell (1991) later extended this to three-species food chains, showing chaos is possible.

These updates, including Rosenzweig's (1971) paradox of enrichment, are now standard for realistic modeling.

Hadziabdic et al. (2017) offered a simpler three-species Lotka-Volterra variant (one prey, two predators). They found three equilibria: if prey hits zero, both predators die, if both predators are zero, prey grows without bound.

Kwagkhor et al. (2024) reworked that model into a trophic cascade: Lions (top predator), Leopards (mid-predator/prey), and Hares (base prey). Lions eat both leopards and hares; leopards eat hares; hares eat plants. Recent three-species food web models with logistic growth and Holling Type II functional responses have successfully analyzed stability, bifurcation, and coexistence in similar intraguild systems (Kamal & Mustafa, 2025).

Empirical studies show that lions influence leopards distribution, behavior, and population dynamics through direct killing and interference, although coexistence without demographic suppression is possible under certain conditions. Hares serve as a shared prey resource, linking the lower trophic level to both carnivores and the ecosystem, contributing to ecosystem stability. (Balme et al., 2017; Miller et al., 2018; Everatt et al., 2019).

Our work refines Kwaghkor et al.'s (2024) lion-leopard-hare setup to boost its ecological realism and analytical depth.

The three-species predator-prey model developed by Kwaghkor et al. (2024) for lions(x), leopards (y), and hares (z) represents a valuable initial attempt to capture

intraguild predation in the African context. The system is given by

$$\frac{dx}{dt} = \alpha x + \delta xy + \rho xz - \sigma x \quad (1)$$

$$\frac{dy}{dt} = \beta y + \tau yz - \varphi xy - \kappa y \quad (2)$$

$$\frac{dz}{dt} = \gamma z - \mu xz - \varepsilon yz - \xi z \quad (3)$$

where $x(0) = x_0, y(0) = y_0$ *and* $z(0) = z_0$

However, this foundation has clear flaws. Without logistic terms, populations grow without bound or swing endlessly, ignoring real carrying capacities. The linear (Type I) feeding ignores handling time and saturation, driving unrealistic booms or crashes at high prey densities. And without density-dependence, the model falls short for guiding conservation in resource-limited settings like savannas.

To address this, we modify the work of Kwaghkor et al. (2024) by adding logistic growth and Holling Type II functional responses across all species. Our specific objectives are: (1) reformulate the lion–leopard–hare model accordingly, (2) confirm positivity and boundedness of solutions and identify some biologically feasible equilibria, (3) perform local stability analysis via the Jacobian matrix and Routh-Hurwitz criterion, and (4) run numerical simulations to illustrate the dynamics and validate our results.

This study provides a more realistic three-species model for studying intraguild predation chains. The stability thresholds we derive via Routh-Hurwitz give wildlife managers clear parameter guidelines for predicting intervention outcomes. We keep the scope focused on a deterministic autonomous system of three nonlinear ODEs, restricted to positivity, boundedness, equilibrium existence, and local asymptotic stability.

MATERIALS AND METHODS

To develop a mathematical model that describes the relationship between prey and predators few assumptions are made as presented below

Assumptions of the Model

i. The first species is a predator, the second species is a prey to the first species and a predator to the third species, while the third species is a prey to both the first and second species.
ii. The first species only dies from natural causes
iii. The second predator/first prey only dies if eaten by the first predator or of natural causes
iv. The second prey only dies if eaten by the first predator, the second predator or natural causes
v. Population birth and death rates are proportional to the population size.
vi. The model equations developed describe the dynamics of a biological system of nonlinear differential equations, which focuses on prey-predator interactions and ignores immigration, emigration, or disease.
vii. The interactions between predators and prey are described by functions.

Table 1: Variables and parameters of the model

Variable/Parameters	Descriptions
$x(t)$	Population of the Lions at time t
$y(t)$	Population of the Leopard at time t
$z(t)$	Population of the hare at time t
α	The growth rate of lion
σ	The natural death rate of lion
δ	Lion benefit from leopard
ρ	Lion benefit from hare
β	The growth rate of leopard
κ	The natural death rate of leopard
τ	Leopard benefit from hare
φ	Lion predation on leopard
γ	The growth rate of hare
ξ	The natural death rate of hare
μ	Lion predation on hare
ε	Leopard predation on hare
k_x	Lion carrying capacity
k_y	Leopard carrying capacity
k_z	Hare carrying capacity
$h_1, \ldots h_6$	The handling times

The original model is recovered when $k_x, k_y, k_z \to \infty$ and $h_i \to 0$

Model Equations

The modified intraguild three-species predator-prey model incorporating logistic growth for all populations and Holling type II functional responses to all predation terms is given by the following system of non-linear ordinary differential equations:

$$\frac{dx}{dt} = \alpha x\left(1 - \frac{x}{k_x}\right) + \frac{\delta xy}{1+h_1 y} + \frac{\rho xz}{1+h_2 z} - \sigma x \quad (4)$$

$$\frac{dy}{dt} = \beta y\left(1 - \frac{y}{k_y}\right) + \frac{\tau yz}{1+h_3 z} - \frac{\varphi xy}{1+h_4 y} - \kappa y \quad (5)$$

$$\frac{dz}{dt} = \gamma z\left(1 - \frac{z}{k_z}\right) - \frac{\mu xz}{(1+h_5 z)} - \frac{\varepsilon yz}{1+h_6 z} - \xi z \quad (6)$$

$where\ x(0) = x_0, y(0) = y_0, \quad and\ z(0) = z_0$

Description of the Model

The model tracks three populations, Lions, leopards and hares. For lions (x), the net rate combines positive birth rate αx, gains from consuming leopards δxy and hares ρxz, and losses from natural death $-\sigma x$, all under carrying capacity k_x. For leopards (y), births βy and hare consumption τyz add to growth, while natural mortality $-\kappa y$ and lion predation $-\varphi xy$ subtract, capped by the carrying capacity k_x . Hares (z) increase through births γz but decline due to natural death $-\xi z$ and predation by both lions $-\mu xz$ and leopards $-\varepsilon yz$,

with k_z as their carrying capacity. All coefficients $(\alpha, \delta, \rho, \sigma, \beta, \tau, \varphi, \kappa, \mu, \gamma, \varepsilon, \xi)$ are non-negative constants while the $h_i's$ are handling times.

Positivity of the solution

Establishing non-negativity and a positively feasible region is an important step in nonlinear population models, because it ensures that the mathematical solutions remain biologically meaningful. A similar proof of positivity and feasibility was presented by Karma et al. (2026). Thus, for the model equations (4), (5), and (6) to have biological meaning, the solutions x(t), y(t), and z(t) must all be positive. Therefore, we prove the positivity of the solution of the model as follows:

Corollary 1

Let the non-negative orthant $\Omega = \{(x, y, z) \epsilon \mathbb{R}_+^3 : x \geq 0, y \geq 0, z \geq 0\}$ be a positive invariant for the model equations with the solution $x(t) > 0, y(t) > 0, and\ z(t) > 0\ \forall\ t \geq 0$ all positive with initial conditions $x(0) \geq 0, y(0) \geq 0, and\ z(0) \geq 0$.

Proof: we shall consider the model equations (4), (5), and (6) as follows

Since each equation contains the population variable as a factor, we can factorize the population variables as follows:

$$\frac{dx}{dt} = x\left[\alpha\left(1 - \frac{x}{k_x}\right) + \frac{\delta y}{1+h_1 y} + \frac{\rho z}{1+h_2 z} - \sigma\right] \quad (7)$$

$$\frac{dy}{dt} = y\left[\beta\left(1 - \frac{y}{k_y}\right) + \frac{\tau z}{1+h_3 z} - \frac{\varphi x}{1+h_4 y} - \kappa\right] \quad (8)$$

$$\frac{dz}{dt} = z\left[\gamma\left(1 - \frac{z}{k_z}\right) - \frac{\mu x}{(1+h_5 z)} - \frac{\varepsilon y}{1+h_6 z} - \xi\right] \quad (9)$$

Let $\{x(t), y(t), z(t)\}$ be a solution of system (4) – (6) With initial conditions $x(0) \geq 0, y(0) \geq 0, and\ z\ (0)0. x(t) \geq, y(t) \geq 0, z(t) \geq 0\ \forall\ t \geq 0.$

We examine the system's behavior on each coordinate plane.

i. Plane $x = 0$

Putting $x = 0$ on (7) we have $\left.\frac{dx}{dt}\right|_{x=0} = 0$

ii. Plane $y = 0$

Putting $y = 0$ in (8) ,we have $\left.\frac{dy}{dt}\right|_{y=0} = 0$

iii. Plane $z = 0$

Putting $z = 0$ in (9) ,we have $\left.\frac{dz}{dt}\right|_{z=0} = 0$

Since each boundary hyperplane $x = 0, y = 0\ and\ z = 0$ the corresponding derivative is zero, it follows that tracjectories starting in $\mathbb{R}_+^3$ cannot cross into the negative region.

Thus$\{x(0), y(0), z(0)\} \in \Omega \implies \{x(t), y(t), z(t)\} \in \Omega\ \forall\ t \geq 0.$

Hence, the non-negative orthant Ω is positively invariant for the system.

Boundedness of solutions

Theorem 1 (Logistic Comparison Theorem)

Suppose $x(t)$ satisfy

$\frac{dx}{dt} \leq rx\left(1 - \frac{x}{k}\right)\ with\ x(0) = x_0 \geq 0, r > 0, and\ k > 0$. Then

$x(t) \leq max(x_0, k)\ \forall\ t \geq 0.$ Brauer,F., & Castillo-Chavez, C. (2012); Murray, J.D (2002).

Lemma 1

Let all the solutions of the systems in the Ω that start in be ultimately bounded. That is there exist a constant $M > 0$ such that $\lim_{t\to\infty} Sup(x(t) + y(t) + z(t)) \leq M$

Proof.

We apply the standard comparison method using the logistic terms.

For the lion population $x(t)$: $\frac{dx}{dt} \leq x\left[\alpha\left(1 - \frac{x}{k_x}\right) + \delta + \rho - \sigma\right]$ (10)

$= x\left[\alpha + \delta + \rho - \sigma - \frac{\alpha x}{k_x}\right].$

Let $A = \alpha + \delta + \rho - \sigma$ and $B = \frac{\alpha}{k_x} > 0$. Then $\frac{dx}{dt} \leq Ax - Bx^2$.

By the standard logistic comparison theorem, $x(t) \leq max\left(x(0), \frac{A}{B}\right)\ \forall\ t \geq 0.$

Hence, x(t) is ultimately bounded above by $M_x = \frac{A}{B} +$ 1. For the leopard population $y(t)$: $\frac{dy}{dt} \leq y\left[\beta\left(1 - \frac{y}{k_y}\right) + \tau - \kappa\right] = y\left[\beta + \tau - \kappa - \frac{\beta y}{k_y}\right]$. (11)

Let $C = \beta + \tau - \kappa\ and\ D = \frac{\beta}{k_y} > 0$, then $y(t) \leq max\left(y(0), \frac{C}{D}\right)\ \forall\ t \geq 0.$ Thus, y(t) is ultimately bounded above by $M_y = \frac{C}{D} + 1.$

2. For the Leopard population $z(t)$: $\frac{dz}{dt} \leq z\left[\gamma\left(1 - \frac{z}{k_z}\right) - \xi\right] = z\left[\gamma - \xi - \frac{\gamma z}{k_z}\right]$ (12)

Let $E = \gamma - \xi\ and\ F = \frac{\gamma}{k_z} > 0.$

Then$z(t) max\left(z(0), \frac{E}{F}\right)$. Hence, z(t) is ultimately bounded above by $M_z = \frac{E}{F} + 1.$

Adding the three inequalities and taking the $\lim_{t\to\infty} sup$ yields

$\lim_{t\to\infty} sup\ (x(t) + y(t) + z(t)) \leq M_x + M_y + M_z = M < \infty.$ (13)

These establish the ultimate boundedness of all the solutions. Therefore, the system is dissipating and all trajectories eventually enter a compact attracting set contained in.

Biologically feasible equilibrium points and conditions for their existence.

The equilibrium points of the system are obtained by setting the right-hand side of the equations (7), (8) and (9) simultaneously to zero as follows:

$$x\left[\alpha\left(1 - \frac{x}{k_x}\right) + \frac{\delta y}{1+h_1 y} + \frac{\rho z}{1+h_2 z} - \sigma\right] = 0 \quad (14)$$

$$y\left[\beta\left(1 - \frac{y}{k_y}\right) + \frac{\tau z}{1+h_3 z} - \frac{\varphi x}{1+h_4 y} - \kappa\right] = 0 \quad (15)$$

$$z\left[\gamma\left(1 - \frac{z}{k_z}\right) - \frac{\mu x}{(1+h_5 z)} - \frac{\varepsilon y}{1+h_6 z} - \xi\right] = 0 \quad (16)$$

We will analyze some of the possible combination and derive the conditions under which the resulting equilibria lie in the non-negative orthant.

$\Omega = \{(x, y, z) \epsilon \mathbb{R}_+^3 : x \geq 0, y \geq 0, z \geq 0\}.$

Trivial Equilibrium

E_0 (0,0,0) this point always exists and corresponds to the complete extinction of all three species.

Single-species Equilibria

Case 1: Lion only equilibria$(x > 0, y = 0, z = 0)$. Putting y = 0 and z= 0 in equations (14), (15) and (16) we have $\alpha x\left(1 - \frac{x}{k_x}\right) - \sigma x = 0$ (17)

$\Rightarrow x\left[\alpha\left(1 - \frac{x}{k_x}\right) - \sigma\right] = 0, since\ x > 0$

We have $\alpha\left(1 - \frac{x}{k_x}\right) - \sigma = 0$, Simplifying for x we have $x_1 = k_x\left(1 - \frac{\sigma}{\alpha}\right)$.

Thus, $E_x = (x_1, 0,0), where\ x_1 = k_x\left(1 - \frac{\sigma}{\alpha}\right)$. This equilibrium is biologically feasible if and only if $\alpha > \sigma$.

Case 2: Leopard-only equilibrium $(x = 0, y > 0, z = 0)$ Putting x = 0 and z= 0 in (14), (15) and (16) we have$\beta\left(1 - \frac{y}{k_y}\right) - \kappa = 0$ (18)

Making y the subject we have$y_1 = k_y\left(1 - \frac{\kappa}{\beta}\right)$. Thus $E_y = (0, y_1, 0), where\ y_1 = k_y\left(1 - \frac{\kappa}{\beta}\right)$. This equilibrium is biologically feasible if and only if $\beta > \kappa$.

Case 3: Hare-only equilibrium um E_z $(x = 0, y = 0, z > 0)$. Putting x = 0 and y= 0 in (14), (15) and (16) we have $\gamma\left(1 - \frac{z}{k_z}\right) - \xi = 0$ (19)

Making z the subject we have $z_1 = k_z\left(1 - \frac{\xi}{\gamma}\right)$. Thus $E_z = (0,0, z_1), where\ z_1 = k_z\left(1 - \frac{\xi}{\gamma}\right)$.

This equilibrium is biologically feasible if and only if $\gamma > \xi$.

Full Interior Equilibrium E_{xyz} $(x > 0, y > 0, z > 0)$.

Dividing (14) by x we have

$\alpha\left(1 - \frac{x}{k_x}\right) + \frac{\delta y}{1+h_1 y} + \frac{\rho z}{1+h_2 z} = \sigma$ (20)

Making x the subject in (20) we have

$x^* = k_x\left[1 - \frac{\sigma}{\alpha} + \frac{\delta y}{\alpha(1+h_1 y)} + \frac{\rho z}{\alpha(1+h_2 z)}\right]$ (21)

Making y the subject in (15) we have

$y^* = k_y\left[1 - \frac{\kappa}{\beta} + \frac{\tau z}{\beta(1+h_3 z)} - \frac{\varphi x}{\beta(1+h_4 y)}\right]$ (22)

Making z the subject of (16) we have

$z^* = k_z\left[1 - \frac{\xi}{\gamma} - \frac{\mu x}{\gamma(1+h_5 z)} - \frac{\varepsilon y}{\gamma(1+h_6 z)}\right]$ (23)

Therefore $E^* = (x^*, y^*, z^*)$. This equilibrium is biologically feasible if $x^* > 0 \iff \alpha > \sigma, y^* > 0 \iff \beta > \kappa, \frac{\tau z}{\beta(1+h_3 z)} > \frac{\varphi x}{\beta(1+h_4 y)}, z^* > 0 \iff \left[1 - \frac{\xi}{\gamma} - \frac{\mu x}{\gamma(1+h_5 z)} - \frac{\varepsilon y}{\gamma(1+h_6 z)}\right] > 0$

Table 2: Summary of the Equilibria

Equilibria	Point	Existence conditions
E_0	(0,0,0)	Always
E_x	$(x_1, 0,0)$	$\alpha > \sigma$
E_x	$(0, y_1, 0)$	$\beta > \kappa$
E_y	$(0,0, z_1)$	$\gamma > \xi$
E^*	(x^*, y^*, z^*)	$x^*, y^*, z^* > 0$

Local asymptotic stability analysis

To perform local stability analysis, we evaluate the Jacobian matrix of the system at each equilibrium points and then we apply the Routh-Hurwitz criterion. An equilibrium is locally asymptotically stable if all the eigenvalues have negative real parts. The characteristic polynomial of the Jacobian is cubic which is

$\lambda^3 + a\lambda^2 + b\lambda + c = 0$ (24)

So the stability is established using Routh-Hurwitz criterion which are $a > 0, c > 0\ and\ ab > c$. Denoting (14), (15) and (16) by $f, g\ and\ h$ we have

$f(x,y,z) = \alpha x\left(1 - \frac{x}{k_x}\right) + \frac{\delta xy}{1+h_1 y} + \frac{\rho xz}{1+h_2 z} - \sigma x$ (25)

$g(x,y,z) = \beta y\left(1 - \frac{y}{k_y}\right) + \frac{\tau yz}{1+h_3 z} - \frac{\varphi xy}{1+h_4 y} - \kappa y$ (26)

$h(x,y,z) = \gamma z\left(1 - \frac{z}{k_z}\right) - \frac{\mu xz}{(1+h_5 z)} - \frac{\varepsilon yz}{1+h_6 z} - \xi z$ (27)

Thus, the calculated Jacobian is:

$$J_{(x,y,z)} = \begin{pmatrix} \alpha\left(1 - \frac{2x}{k_x}\right) + \frac{\delta y}{1+h_1 y} + \frac{\rho xz}{1+h_2 z} - \sigma & \frac{\delta x}{(1+h_1 y)^2} & \frac{\rho x}{(1+h_2 z)^2} \\ -\frac{\varphi y}{1+h_4 y} & \beta\left(1 - \frac{2y}{k_y}\right) + \frac{\tau z}{1+h_3 z} - \frac{\varphi x}{(1+h_4 y)^2} - \kappa & \frac{\tau y}{(1+h_3 z)^2} \\ -\frac{\mu z}{1+h_5 z} & -\frac{\varepsilon z}{1+h_6 z} & \gamma\left(1 - \frac{2z}{k_z}\right) + \frac{\mu x}{(1+h_5 z)^2} + \frac{\varepsilon y}{(1+h_6 z)^2} - \xi \end{pmatrix} \quad (28)$$

We now apply the Routh-Hurwitz criterion at each biologically feasible equilibrium.

Stability of Trivial Equilibrium

Calculating the Jacobian at $E_0 = (0,0,0)$ we have Thus, the Jacobian is

$$J_{(0,0,0)} = \begin{pmatrix} \alpha - \sigma & 0 & 0 \\ 0 & \beta - \kappa & 0 \\ 0 & 0 & \gamma - \xi \end{pmatrix} \quad (29)$$

The Jocobian is a diagonal matrix, hence, the eigenvalues are $\lambda_1 = \alpha - \sigma, \lambda_2 = \beta - \kappa, and\ \lambda_3 = \gamma - \xi$. Therefore, E_0 is locally asymptotically stable if $\alpha < \sigma, \beta < \kappa, and\ \gamma < \xi$, otherwise it is not stable.

Stability of Single-Species Equilibria

Lion only equilibria that is $E_x = (x_1, 0,0)\ where\ x_1 = k_x\left(1 - \frac{\sigma}{\alpha}\right) > 0$. Calculating the Jacobian at $E_x = (x_1, 0,0)$ we have

$$J_{E_x} = \begin{pmatrix} \alpha - \sigma & \delta x_1 & \rho x_1 \\ 0 & \beta - \kappa - \varphi x_1 & 0 \\ 0 & 0 & \gamma - \xi - \mu x_1 \end{pmatrix} \quad (30)$$

The Jacobian is block-triangular, so the eigenvalues are $\lambda_1 = -\frac{\alpha x_1}{k_x} < 0\ since\ x_1 > 0$, $\lambda_2 = \beta - \kappa - \varphi x_1$,

$\lambda_3 = \gamma - \xi - \mu x_1$. Thus, E_x is locally asymptotically stable if and only if $\varphi x_1 > \beta - \kappa$ and $\mu x_1 > \gamma - \xi$. Otherwise it is unstable.

Leopard only: $E_y = (0, y_1, 0)$ *where* $y_1 = y\left(1 - \frac{\kappa}{\beta}\right) > 0$ *assuming* $\beta > \kappa$ at $E_y, g(0, y_1, 0) = 0$ the Jacobian is:

$$J_(0, y_1, 0) = \begin{pmatrix} \alpha - \sigma + \frac{\delta y_1}{1+h_1 y_1} & 0 & 0 \\ -\frac{\varphi y_2}{1+h_4 y_1} & \kappa - \beta & \tau y_1 \\ 0 & -\frac{\epsilon y_1}{1+h_6 y_1} & \gamma - \xi - \frac{\varepsilon y_1}{1+h_6 z} \end{pmatrix} \quad (31)$$

The Jacobian is block-triangular, therefore the eigenvalues are are $\lambda_1 = \alpha - \sigma + \frac{\delta y_1}{1+h_1 y_1}$, $\lambda_2 = \kappa - \beta = -\frac{\beta y_1}{k_y} < 0$ and $\lambda_3 = \gamma - \xi - \frac{\varepsilon y_1}{1+h_6 z}$. Hence, E_y is locally asymtotically stable if $\alpha + \frac{\delta y_1}{1+h_1 y_1} < \sigma$ and $\varepsilon y_1 > \gamma - \xi$.

Hare only equilibria $E_z = (0,0,z_1)$ *where* $z_1 = k_z\left(1 - \frac{\xi}{\gamma}\right) > 0$ *assuming* $\gamma > \xi$ at $E_y, G(0,0,z_1) = 0$ the Jacobian is:

$$J_{(E_z)} = \begin{pmatrix} \alpha - \sigma + \frac{\rho z_1}{1+h_2 z_1} & 0 & 0 \\ 0 & \beta - \kappa + \frac{\tau z_1}{1+h_3 z_1} & 0 \\ \frac{-\mu z_1}{1+h_5 z_1} & \frac{-\epsilon z_1}{1+h_6 z_1} & \xi - \gamma \end{pmatrix} \quad (32)$$

The Jacobian is triangular, therefore the eigenvalues are $\lambda_1 = \alpha - \sigma + \frac{\rho z_1}{1+h_1 y_1}$ $\lambda_2 = \beta - \kappa + \frac{\tau z_1}{1+h_3 z_1}$ $\lambda_3 = \xi - \gamma = -\frac{2 z_1}{k_z} < 0$ *assuming* $\gamma > .\xi$. Hence, E_z is locally asymtotically stable if the following condtion are satisfied: $\alpha + \frac{\rho z_1}{1+h_2 z_1} < \sigma$ *and* $\beta + \frac{\tau z_1}{1+h_3 z_1} < \kappa$.

Stability of the Interior Coexistence Equilibrium $\mathbf{E}^* = (\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{z}^*)$

Evaluating the Jacobian at (x^*, y^*, z^*) we have

$$J_{(x^*, y^*, z^*)} = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix} \quad (33)$$

All entries are evaluated at (x^*, y^*, z^*).

Where $J_{11} = \alpha\left(1 - \frac{2x^*}{k_x}\right) + \frac{\delta y^*}{1+h_1 y^*} + \frac{\rho z^*}{1+h_2 z^*} - \sigma$, $J_{12} = \frac{\delta x^*}{(1+h_1 y^*)^2}$, $J_{13} = \frac{\rho x^*}{(1+h_2 z^*)^2}$,

$J_{21} = -\frac{\varphi y^*}{1+h_4 y^*}$, $J_{22} = \beta\left(1 - \frac{2y^*}{k_y}\right) + \frac{\tau z^*}{1+h_3 z^*} - \frac{\varphi x^*}{(1+h_4 y^*)^2} - \kappa$, $J_{23} = \frac{\tau y^*}{(1+h_3 z^*)^2}$

$J_{31} = -\frac{\mu z^*}{1+h_5 z^*}$, $J_{32} = -\frac{\varepsilon z^*}{1+h_6 z^*}$, and $J_{33} = \gamma\left(1 - \frac{2z^*}{k_z}\right) - \frac{\mu x^*}{1+h_5 z^*} - \frac{\varepsilon y^*}{1+h_6 z^*} - \xi$.

The characteristic polynomial is $\lambda^3 + a\lambda^2 + b\lambda + c = 0$, whe re

$a = -\text{trace}(J) = -(J_{11}+J_{22}+J_{33})$, b = Sum of principal 2×2 minors

$= J_{11}J_{22} + J_{11}J_{33} + J_{22}J_{33} - J_{12}J_{21} - J_{13}J_{31} - J_{23}J_{32}$.

And $c = -\det(J)$

Thus, $\mathbf{E}^*$ is locally asymptotically stable if and only if

1. $a > 0 \iff J_{11} + J_{22} + J_{33} < 0$ (34)
2. $c > 0 \iff \det(J) < 0$ (35)
3. $ab - c > 0$ (36)

The (36) above can be expanded to

$-(J_{11}+J_{22}+J_{33})(J_{11}J_{22} + J_{11}J_{33} + J_{22}J_{33} - J_{12}J_{21} - J_{13}J_{31} - J_{23}J_{32}) + \det(J) > 0$ (37)

Where $\det(J) = J_{11}(J_{22}J_{33} - J_{23}J_{32}) - J_{12}(J_{21}J_{33} - J_{23}J_{31}) + J_{13}(J_{21}J_{32} - J_{22}J_{31})$.

RESULTS AND DISCUSSION

In this section we present the numerical simulation of our modified model. The models were solved numerically using MATLAB ODE45 solver to verify the analytical results and to investigate the dynamical behavior of the system under different parameter values, following a similar numerical validation approach reported by Karma et al. (2026). The parameter values used in the simulations were adopted from the work Kwaghkor et al. (2024).

Table 3: Parameter values and initial conditions

Parameter	Scenario 1	Scenario 2	Scenario 3	Scenario 4	Scenario 5
Growth Rates					
α	0.80	0.80	0.80	0.10	0.90
β	0.90	0.10	0.90	0.10	1.00
γ	1.20	1.20	1.20	1.20	1.50
Mutualism					
δ	0.05	0.05	0.05	0.05	0.08
ρ	0.03	0.03	0.03	0.03	0.06
τ	0.02	0.02	0.02	0.02	0.05
Predation					
φ	0.15	0.50	0.60	0.15	0.25
μ	0.20	0.20	0.60	0.20	0.40
ε	0.18	0.18	0.60	0.18	0.35
Death Rates					
σ	0.20	0.20	0.20	0.50	0.20
κ	0.15	0.50	0.15	0.50	0.15
ξ	0.10	0.10	0.10	0.10	0.10
Handling Times					
h_1	0.1	0.1	0.1	0.1	0.1
h_2	0.08	0.08	0.08	0.08	0.08
h_3	0.06	0.06	0.06	0.06	0.06
h_4	0.10	0.10	0.10	0.10	0.10
h_5	0.05	0.05	0.05	0.05	0.05
h_6	0.07	0.07	0.07	0.07	0.07
Carrying Capacities					
k_x	100	100	100	100	100
k_y	120	120	120	120	120
k_z	200	200	200	200	200

NUMERICAL SIMULATIONS

The following are the graphs obtained from numerical simulations using for our modified model.

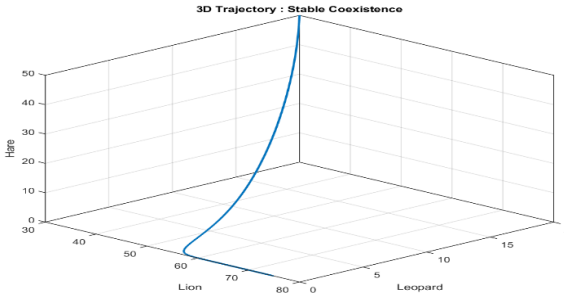


Figure 1: Three-dimensional portrait showing the stable coexistence.

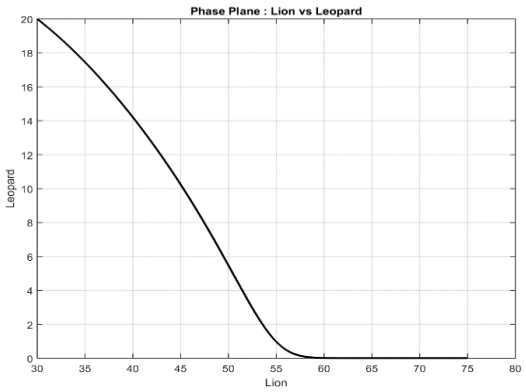


Figure 2: Phase portrait of lion-leopard stable coexistence dynamics

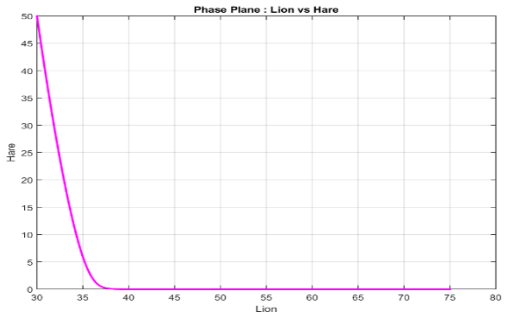


Figure 3: Phase portrait of lion-hare dynamics under stable coexistence.

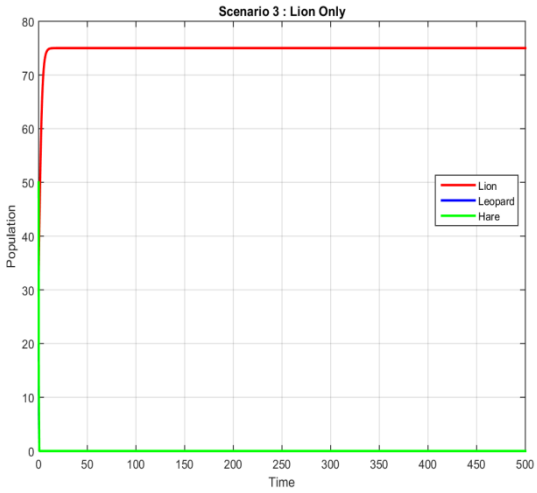


Figure 4: Time series plot showing lion-leopard coexistence.

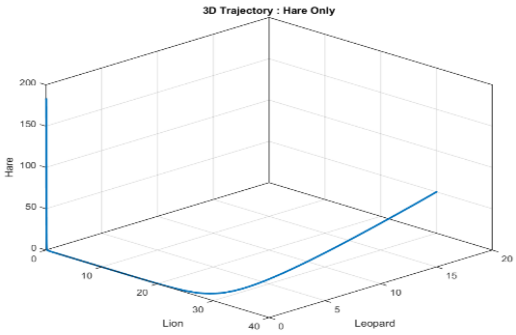


Figure 5: Three-dimensional portrait showing hare only persistence.

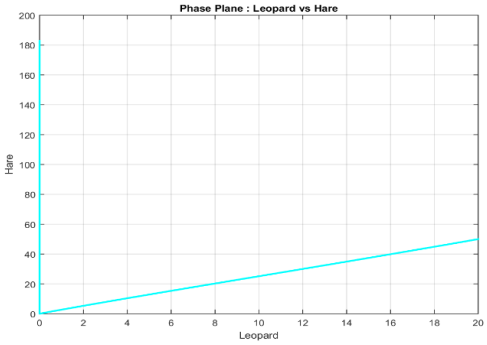


Figure 6: Phase portrait of leopard-hare dynamics under hare-only persistence

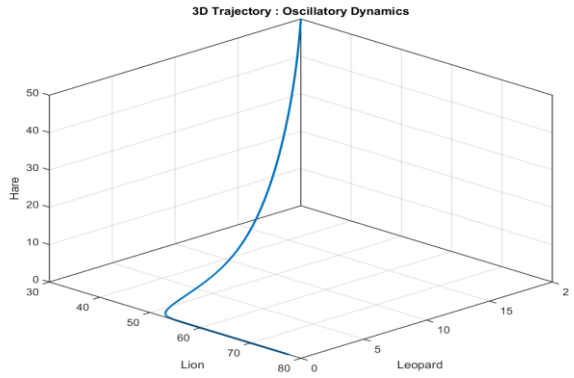


Figure 7: Three-dimensional phase portrait showing sustained oscillatory dynamics.

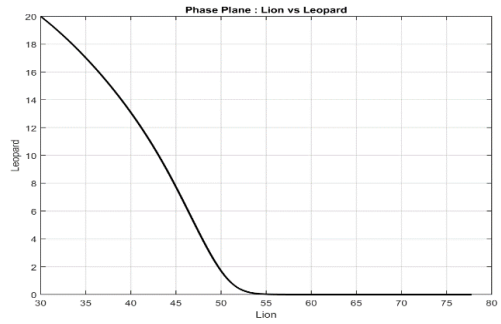


Figure 8: Phase portrait of lion-leopard under oscillatory behavior.

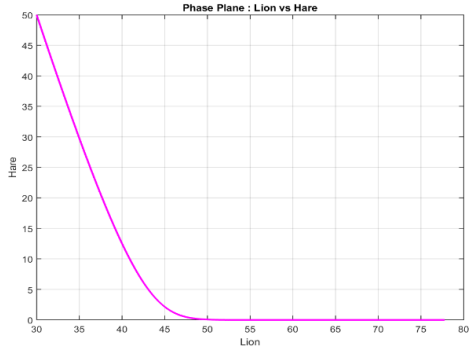


Figure 9: Phase portrait of lion-hare dynamics under oscillatory behavior.

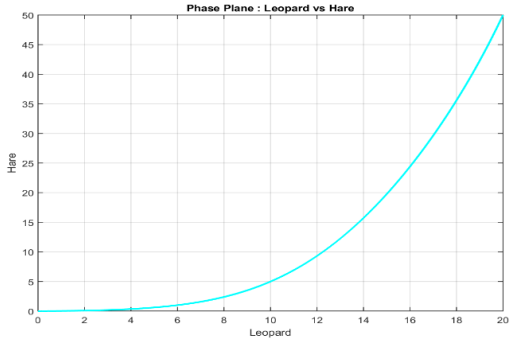


Figure 10: Phase portrait of leopard-hare dynamics under oscillatory behavior.

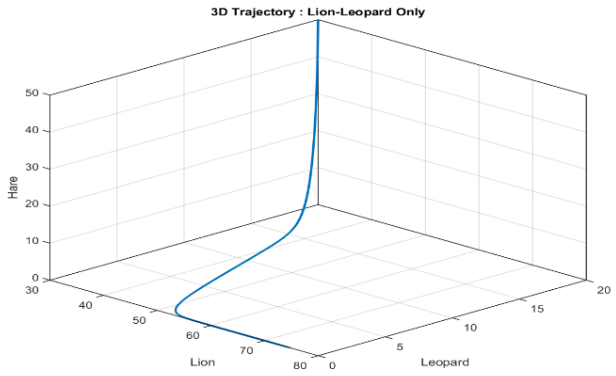


Figure 11: Three-dimensional phase portrait showing lion-leopard coexistence with hare extinction.

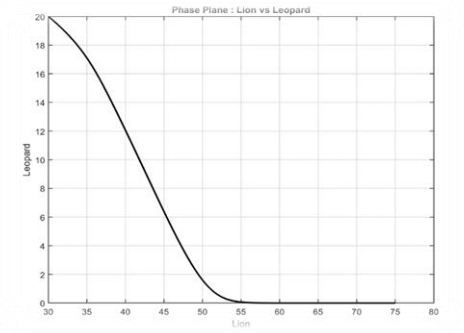


Figure 12: Phase portrait of lion-leopard under oscillatory behavior.

In Scenario one, all three species show damped oscillations that settle at stable equilibria: lions are approximately 55, leopards are approximately 20, hares are approximately 33. This confirms that logistic growth and Holling Type II functional responses support stable multi-species coexistence. The damped pattern suggests a stable focus equilibrium, indicating ecological resilience.

Scenario two yields two-species persistence, with lions are approximately 57 and hares are approximately 31 surviving while leopards go extinct over 50–100 time units. Lion numbers rise slightly due to competitive release, while hares drop a bit from heavier lion predation—highlighting indirect interactions in intrag uild predation systems. The gradual decline, unlike abrupt extinctions in the original model, reflects density-dependent slowing.

In Scenario three, lions grow logistically to carrying capacity (100), while leopards and hares crash quickly. The S-shaped growth contrasts with the original's unbounded exponential rise. Hares vanish under high predation ($\mu = 0.6$, $\varepsilon = 0.6$), and leopards follow as a secondary cascade—showing how simple food webs are vulnerable to cascading extinctions.

Scenario four shows hares growing logistically to capacity (200), while both predators decline to extinction. The smooth approach to capacity reflects density-dependent regulation in a prey population freed from predation. Predators fail due to low growth rates ($\alpha = 0.10$, $\beta = 0.10$) and high death rates ($\sigma = 0.50$,

$\kappa = 0.50$), signaling an unsuitable environment for predators.

The final scenario presents the richest dynamics: sustained oscillations—lions 35–70, leopards 10–25, hares 15–45. Unlike Scenario one's damped cycles, these persist throughout, indicating a stable limit cycle with classic predator-prey phase lags, consistent with field observations.

Overall, the original model (linear terms, unbounded growth) produces unrealistic dynamics—no stable coexistence, abrupt extinctions, or erratic swings. The revised model (logistic growth, Holling Type II) generates ecologically meaningful outcomes: stable equilibria, gradual extinctions, and patterns that align with empirical data, confirming its value for applied ecology.

CONCLUSION

This study examines the population dynamics of a three-species intraguild predator-prey system involving lions (x), leopards (y), and hares (z). The work addresses limitations in the linear model of Kwaghkor et al. (2024), The model is reformulated by incorporating logistic growth terms for all three populations and replacing linear Type I responses with saturating Holling Type II functional responses, resulting in a system of three nonlinear ordinary differential equations.

Positivity and boundedness of solutions are rigorously proved. Some biologically feasible equilibria are located with explicit stability conditions derived via the Jacobian matrix and Routh–Hurwitz criterion, yielding biologically interpretable parameter thresholds.

The numerical simulations performed shows that the work of Kwaghkor et, al. (2024) linear model produces persistent large-amplitude oscillations, while the modified model exhibits damped convergence or monotonic approach to stable equilibria. The study contributes a more ecologically realistic three-species model suitable for studying intraguild predation chains, with stability conditions providing clear thresholds for conservation and wildlife management applications.

This study demonstrates that incorporating logistic growth and Holling Type II functional responses transforms the unrealistic oscillatory model of Kwaghkor et al. (2024) into a biologically meaningful and ecologically realistic framework for intraguild predation. Density-dependent self-limitation and predator saturation are essential for eliminating unrealistic behaviors and generating stable equilibria. These findings bridge a gap in the mathematical biology literature and offer practical parameter-based thresholds for conservation decisions. The results affirm that modest biological refinements yield qualitatively superior predictions, establishing the modified model as a valuable tool for both theoretical analysis and applied wildlife management.

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