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Mathematical Modelling of the Transmission Dynamics of Hepatitis B Incorporating Vertical and Horizontal Transmission with Vaccination and Treatment as Control Strategies

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ABSTRACT

In this study, we formulated a mathematical model of the transmission dynamics of hepatitis B considering both vertical and horizontal transmission and the impact of treatment and vaccination as control measures. The basic reproduction number R_0^B was calculated to represent the persistence and eradication conditions of the disease. The following results of the analysis indicate that the disease free equilibrium is locally stable for $R_0^B < 1$, globally stable for $R_0^B < 1$ and a stable endemic equilibrium is reached for $R_0^B > 1$. The parameters whose values had a positive sensitivity index increased the transmission of hepatitis B, such as contact and transmission rate, while the parameters whose values had a negative sensitivity index reduced the transmission of hepatitis B, such as the treatment and vaccination rate, etc. Numerical simulations also revealed that if the treatment and vaccination rate is increased, the burden of hepatitis B to the population is greatly reduced. This underscores the need for treatment interventions to break and potentially prevent the transmission of hepatitis B in humans. Thus, it is recommended to improve vaccination and treatment programmes, including interventions to prevent mother-to-child transmission, to effectively reduce and control the transmission of hepatitis B to the population.

Keywords:
Hepatitis B;
Vaccination;
Vertical transmission;
Horizontal transmission

INTRODUCTION

Hepatitis B remains a major infectious disease of global public health concern, posing a significant threat to human health and survival worldwide. Hepatitis B is caused by highly infectious hepatitis B virus (HBV) which mainly affects the liver and can lead to acute and chronic disease. Chronic HBV infection can lead to cirrhosis, liver failure and HCC. Despite the fact that the disease is preventable by vaccination, and can be managed through effective antiviral medications, hepatitis B remains a significant global public health problem. According to the latest estimates from the World Health Organization, there were an estimated 240 million people with chronic HBV infection in 2024, and an estimated 0.9 million new HBV infections in 2024. Furthermore, an estimated 1.1 million people died from cirrhosis and liver cancer associated with chronic hepatitis B virus (HBV) infection in 2024 (WHO, 2026).

These statistics illustrate that although significant strides have been made in the prevention and treatment of HBV infections, the disease is still an important preventable cause of mortality and morbidity around the world.

HBV burden is extremely high in the African Region. In 2024, the region had an estimated 68% of the global burden of new hepatitis B infections and coverage of the birth dose vaccine against hepatitis B was low in many countries. Moreover, less than 5% of all people with chronic HBV worldwide are in treatment in 2024 (WHO, 2026). Nigeria has a large burden of HBV and is one of the countries that continue to contribute significantly to the global burden of HBV. The continuing burden of HBV in Nigeria and other African countries highlights the need for effective combinations of preventive and therapeutic interventions. HBV can be transmitted from mother to child, via sexual contact, contact with infected blood and body fluids,

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contaminated blood products from transfusion, contaminated healthcare items such as inadequate sterilisation of healthcare items, infected organs (Moneim et al., 2009; Nguyen et al., 2020). There are two types of transmission: vertical transmission and horizontal transmission. Vertical transmission can happen when an infected mother passes on the virus to her child in utero, at birth, or within the first month after birth. This is especially significant since the chances of developing a chronic infection following an early childhood infection is high. Horizontal transmission is via contact with infected blood or body fluids, such as sexual contact, contact with contaminated instruments or blood products. Hence, the vertical and horizontal transmission should be included in a mathematical model to properly describe the transmission dynamics of HBV.

The outcome of HBV infection is dependent on the interaction with the host immune system. Once a person has been exposed to HBV, they can successfully eliminate the virus and become immune or may not be able to eradicate it and develop chronic infection. So the host immune response is a significant factor in viral persistence and disease progression (Busca & Kumar, 2014). The biological features of HBV transmission are appropriate for mathematical modelling as people can be in various epidemiological states, depending on their infection status, disease course, treatment and recovery. Vaccination is still one of the most effective means of preventing HBV infection. The hepatitis B birth dose is particularly important for preventing mother-to-child transmission of the disease, if not administered at the proper time. WHO has made hepatitis B vaccination a priority to achieve the target of eliminating viral hepatitis as a public health threat by 2030, including the timely birth-dose vaccination. HBV control is also reliant on treatment, as antiviral drugs can slow the progression of the virus to cirrhosis and liver cancer, thereby decreasing the risk of such outcomes (Nguyen et al., 2020; WHO, 2026). However, the disparity between availability of effective interventions and the actual coverage of these interventions continues to cause significant public health issues especially for HBV transmission. Consequently, study of vaccination and treatment in tandem in a mathematical framework is needed to elucidate how these interventions may affect HBV transmission.

Mathematical modelling is now an important tool to understand the spread of infectious diseases, as well as to evaluate the success of control measures. Mathematical models have proven to be valuable tools for understanding the dynamics of disease, the interventions that can be implemented, and for public health decision-making in recent years. Recently, Jalija et al. (2026) proposed a generalized fractional Adams–Bashforth–Moulton method and discussed the dynamics of typhoid fever, which showed that the fractional numerical methods can be useful and efficient for solving the epidemiological models. With insight into multiple infection dynamics, Odiba et al. (2024) developed a deterministic model of the transmission of COVID-19, HIV and monkeypox, highlighting the need to consider the interactions between each of these diseases when modelling. Furthermore, Ojonimi et al. (2025) presented a fractional-order model for hepatitis B (HB) and applied the generalized Adams–Bashforth–Moulton method, highlighting the accuracy, flexibility, and promise of fractional-order models in the study of intricate dynamics in infectious diseases. Omale et al. (2020) performed sensitivity analysis and stability analysis of the TB transmission and found out the parameters which have the most influence on the propagation of the disease in a population. Similarly, Omale et al. (2021) developed a mathematical model of TB-HIV co-infection in Kogi State, Nigeria and showed that co-infection is an important factor in the dynamics of TB-HIV transmission. Omede et al. (2023) examined the transmission dynamics of the co-infection of HIV and Zika virus (ZIKA) and found that models of disease transmission and control should account for multiple modes of transmission. Amos et al. (2024) developed a fractional mathematical model of transmission dynamics and control of HCV and showed that fractional-order models can better describe the behaviour of a disease than classical integer-order models. Their study also came up with effective control measures that could be taken to control the spread of infection. Bolaji et al. 2024 formulated a mathematical model for controlling the transmission of monkeypox in sub-Saharan Africa and emphasized the significance of preventive and health care measures to limit the transmission of the disease. In addition, Bolaji et al. (2023) found that population density affects the transmission of the Omicron variant of COVID-19 in highly populated urban areas in Nigeria and thus timely interventions are crucial to reduce transmission. Philip et al. (2024) proposed a fractional-order HIV/AIDS transmission model with control measures and demonstrated that the fractional modelling also offers insights into long-term dynamics of HIV/AIDS transmission and effectiveness of intervention measures. Similarly, Amos et al. (2026) formulated a mathematical deterministic model of HIV/AIDS vertical and horizontal transmission and included treatment as a key intervention. They found that higher treatment rates led to lower rates of HIV/AIDS prevalence and new infections, while higher rates of contacts and vertical transmission were linked to higher rates of disease transmission. These findings reinforce the need for a multi-faceted approach, with combination intervention – the use of both treatment and preventive measures for infectious diseases control. Amos et al (2026) formulated a mathematical model of vertical and horizontal transmission dynamics of syphilis, including treatment interventions. Their study showed the necessity to use several modes of transmission and treatment intervention to better understand infectious disease transmission and control. Vertical and horizontal transmission is considered to give a more complete picture of the spread of disease, especially if the vertical transmission and other modes of transmission play a role in the persistence of disease. Agbata et al. (2025) developed a fractional-order mathematical model for the transmission dynamics and control of typhoid fever, incorporating fractional-order derivatives and the Adams--Bashforth numerical method.

Jalija et al. (2025) investigated a fractional-order dengue fever model using the Caputo, Caputo–Fabrizio, and Atangana–Baleanu fractional operators, together with appropriate numerical schemes for analyzing the resulting system.

Specifically, mathematical modelling has been used to the transmission and control of hepatitis B virus. Moneim et al. (2009) used stochastic and Monte Carlo simulation techniques to explore the diffusion of hepatitis B, and Kimbir et al. (2014) developed and simulated a mathematical model of HBV transmission dynamics under the influence of vaccination and treatment to show its significance in controlling HBV transmission. Bowong et al. (2010) developed and solved a mathematical model of HBV-HIV co-infection which included relevant epidemiological and biological features of both diseases. Likewise, Nampala et al., (2018) formulated a mathematical model of hepatotoxicity and antiretroviral therapeutic effects in HIV/HBV co-infection and examined the therapeutic and toxic effects of treatment. Amos et al. (2026) developed a fractional-order Hepatitis B model incorporating vertical and horizontal transmission and control strategies using three fractional operators, highlighting the importance of memory effects in HBV transmission dynamics.

Although these studies have significantly improved the mathematical understanding of hepatitis B virus and other infectious diseases, a complete model for HBV that explicitly considers vertical and horizontal transmission and takes vaccination and treatment as control strategies is still needed. The continuing global burden of HBV, high mortality, low treatment coverage, and the significance of mother-to-child transmission all give strong impetus to further mathematical investigation. The model includes vertical transmission, which is particularly relevant, since infection early in life is associated with a much greater likelihood of chronic infection.

Thus, a deterministic mathematical model of the transmission dynamics of hepatitis B virus is developed in this paper and is rigorously analyzed. The model includes vertical and horizontal transmission, vaccination and treatment as key components of the disease dynamics and control framework. The proposed model is used to evaluate the impact of vaccination and treatment on the transmission of HBV and to identify effective strategies for reducing the spread of HBV within a population. The analysis includes establishment of fundamental properties of the model, determination of disease-free equilibrium and basic reproduction number, stability analysis, sensitivity analysis and numerical simulations.

The results are expected to provide useful mathematical insights into the relative and combined effects of vaccination and treatment on the control of HBV transmission, especially in populations where both vertical and horizontal transmission play a role in disease persistence.

MATERIALS AND METHODS

Model Formulation

In the modeling of Hepatitis B virus, the entire human population is divided into seven categories, using the integer-order model; susceptible individuals S_h, This group includes people who are not infected.; exposed individuals E_B; People who have experienced infection exposure but have not developed contagious status, acute infected individuals I_A, chronic infected individuals I_C, population of individuals on treatment T_B, vaccinated human population V_B and recovered human population R_B.

The recruitment of people into the susceptible group is Λ_h, where are the effective rates of contact of the susceptible people with the acute infected people, the chronic infected population and the population on the hepatitis B treatment are β_1, β_2 *and* β_3 respectively, is the progression rate from exposed human population to acute infected humans with hepatitis B θ_1 and the progression rate from acute infected class with hepatitis B to chronic infected class is ω_1. The acute and chronic infected human population are treated at the rate of ε_6 and ε_1. The rate of recovery of humans due to treatment of hepatitis B is denoted by τ .

The natural death rate of human beings is represented as μ_h . The disease induced death rate of acute infected humans with hepatitis B, chronic infected humans with hepatitis B and humans on treatment of hepatitis B are represented by δ_1, δ_3 *and* δ_6 . The rate at which susceptible humans are vaccinated against hepatitis B is denoted as ϕ_2 and the rate of vaccine failure as ϕ_1 and the rate at which recovered humans become susceptible again is γ. The disease can be transmitted vertically at the rate of β_s.

Hepatitis B Model Flow Diagram

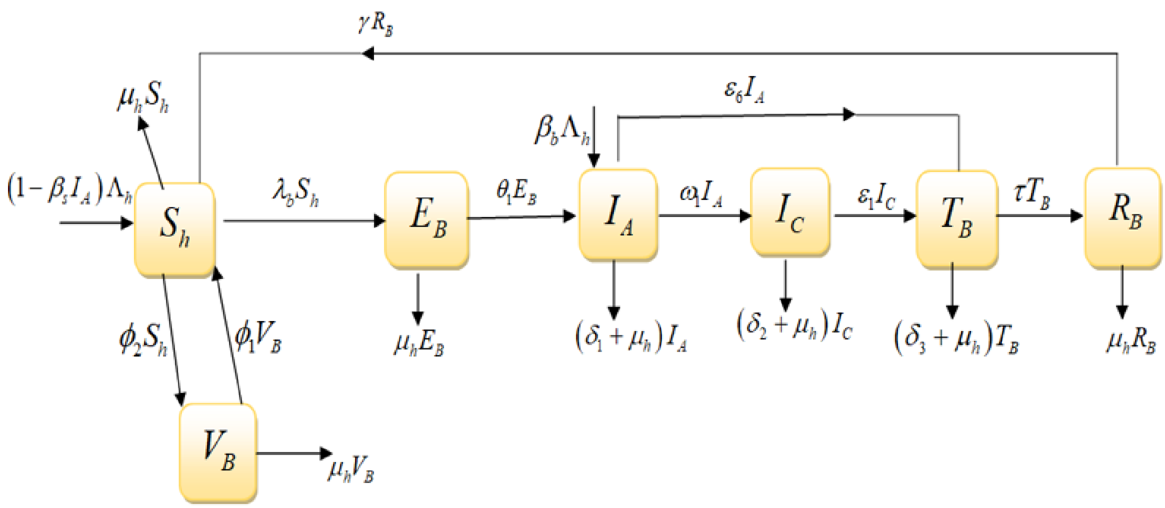

Figure 1: Hepatitis B Model Flow Diagram

Hepatitis B Model equation

$$
\begin{aligned}
\frac{dS_h}{dt} &= (1-\beta_b I_A)\Lambda_h + \phi_1 V_B + \gamma R_B - \lambda_b S_h - (\phi_2 + \mu_h) S_h, \\
\frac{dE_B}{dt} &= \lambda_b S_h - (\theta_1 + \mu_h) E_B, \\
\frac{dI_A}{dt} &= \theta_1 E_B + \beta_b \Lambda_h - (\varepsilon_6 + \omega_1 + \delta_1 + \mu_h) I_A, \\
\frac{dI_C}{dt} &= \omega_1 I_A - (\varepsilon_1 + \delta_3 + \mu_h) I_C, \\
\frac{dT_B}{dt} &= \varepsilon_6 I_A + \varepsilon_1 I_C - (\tau + \delta_6 + \mu_h) T_B, \\
\frac{dV_B}{dt} &= \phi_2 S_h - (\phi_1 + \mu_h) V_B, \\
\frac{dR_B}{dt} &= \tau T_B - (\gamma + \mu_h) R_B.
\end{aligned} \qquad (1)
$$

Where $\lambda_b = \dfrac{(\beta_1 I_A + \beta_2 I_C + \beta_3 T_B)}{N_h}$.

Table 1: Model Variables and Parameters Descriptions

Variables	**Descriptions**
S_h	Humans who are susceptible to hepatitis B
E_B	Humans population who are Exposed to hepatitis B
I_A	Acute infected human infected with hepatitis B
I_C	Chronic infected human infected with hepatitis B
T_B	Human population on hepatitis B treatment
V_B	Vaccinated human population against hepatitis B
R_B	Recovered human population from hepatitis B
Parameters	**Descriptions**
Λ_h	Recruitment rate of Susceptible human population to hepatitis B
μ_h	Natural death rate of human population
β_1	The rate of contact between the susceptible humans and the human acutely infected humans
β_2	Contact rate between the susceptible people and the chronically infected people
β_3	Contact rate between the susceptible human beings and those who are on hepatitis B treatment.
θ_1	The progression rate of exposed humans to Acute infected human population with hepatitis B
ω_1	The rate of progression of the acute infected hepatitis B class to chronic infected hepatitis B class
ε_6	Treatment rate of Acute infected human population.
ε_1	Treatment rate of Chronic infected human population.
τ	The recovery rate of human population due to treatment of hepatitis B.
δ_1	Disease induced death rate of acute Infected humans with hepatitis B
δ_3	Disease induced death rate of chronic Infected humans with hepatitis B
δ_6	The death rate of humans on hepatitis B medication as a result of disease in human population.
ϕ_2	Vaccination rate of human population.

ϕ_1	Waning rate of vaccine
γ	Rate at which recovered humans become susceptible again.
β_s	The rate at which the disease is transmitted vertically from mother to child during pregnancy.

MODEL ANALYSIS

Positivity of Solution of hepatitis B

Theorem 1: If the initial value

$S_h \geq 0, E_B \geq 0, I_A \geq 0, I_C \geq 0, T_B \geq 0, V_B \geq 0, R_B \geq 0.$

Then the solution of hepatitis B model is non-negative.

Proof: To show that solution of the model are positive, we first dS_h/dt from equation

$$\frac{dS_h}{dt} = (1-\beta_b I_A)\Lambda_h + \phi_1 V_B + \gamma R_B - \frac{(\beta_1 I_A + \beta_2 I_C + \beta_3 T_B)}{N_h} S_h - (\phi_2 + \mu_h) S_h, \quad (2)$$

$$\frac{dS_h}{dt} \geq -\frac{(\beta_1 I_A + \beta_2 I_C + \beta_3 T_B + \phi_2 + \mu_h)}{N_h} S_h,$$

Separating the variables, we have:

$$\frac{dS}{S_h} \geq -\frac{(\beta_1 I_A + \beta_2 I_C + \beta_3 T_B + \phi_2 + \mu_h)dt}{N_h},$$

Integrating both sides, we have:

$$\ln S_h \geq -\frac{(\beta_1 I_A + \beta_2 I_C + \beta_3 T_B + \phi_2 + \mu_h)t}{N_h} + C, \quad (3)$$

Taking the exponential of both sides, we have:

$$e^{\ln S_h} \geq e^{-\frac{(\beta_1 I_A + \beta_2 I_C + \beta_3 T_B + \phi_2 + \mu_h)t}{N_h} + \ln C},$$

$$e^{\ln S_h} \geq e - \frac{(\beta_1 I_A + \beta_2 I_C + \beta_3 T_B + \phi_2 + \mu_h)t}{N_h} \times e^{\ln C},$$

$$S_h(t) \geq Ce^{-\frac{(\beta_1 I_A + \beta_2 I_C + \beta_3 T_B + \phi_2 + \mu_h)t}{N_h}},$$

By applying the initial condition t=0 we have:

$S_h(0) \geq C,$

$$S_h(t) \geq S_h(0) e^{-\frac{(\beta_1 I_A + \beta_2 I_C + \beta_3 T_B + \phi_2 + \mu_h)t}{N_h}}. \quad (4)$$

$$^{-\frac{(\beta_1 I_A + \beta_2 I_C + \beta_3 T_B + \phi_2 + \mu_h)t}{N_h}} > 0.$$

Since

For the exposed class $(E_B(t))$, from

$$\frac{dE_B}{dt} = \frac{(\beta_1 I_A + \beta_2 I_C + \beta_3 T_B)}{N_h} S_h - (\theta_1 + \mu_h) E_B, \quad (5)$$

it follows that,

$$\frac{dE_B}{dt} \geq -(\theta_1 + \mu_h) E_B,$$

Hence,

$$E_B(t) \geq E_B(0) e^{-(\theta_1 + \mu_h)t} > 0. \quad (6)$$

For the infectious class $(I_{I_A}(t))$,from

$$\frac{dI_A}{dt} = -(\varepsilon_6 + \omega_1 + \delta_1 + \mu_h) I_A, \quad (7)$$

we have:

$$\frac{dI_A}{I_A} = \theta_1 E_B + \beta_b \Lambda_h - (\varepsilon_6 + \omega_1 + \delta_1 + \mu_h) dt,$$

Therefore,

$$I_A(t) \geq I_A(0) e^{-(\varepsilon_6 + \omega_1 + \delta_1 + \mu_h)t} > 0. \quad (8)$$

For the infectious class $(I_C(t))$,from

$$\frac{dI_C}{dt} = \omega_1 I_A - (\varepsilon_1 + \delta_3 + \mu_h) I_C, \quad (9)$$

we have:

$$\frac{dI_A}{I_A} = \theta_1 E_B + \beta_b \Lambda_h - (\varepsilon_6 + \omega_1 + \delta_1 + \mu_h) dt,$$

Therefore,

$$I_C(t) \geq I_C(0) e^{-(\varepsilon_6 + \omega_1 + \delta_1 + \mu_h)t} > 0. \quad (10)$$

For the treatment class $(T_B(t))$, from

$$\frac{dT_B}{dt} = \varepsilon_6 I_A + \varepsilon_1 I_C - (\tau + \delta_6 + \mu_h) T_B, \quad (11)$$

it follows that,

$$\frac{dT_B}{dt} \geq -\left(\tau + \delta_6 + \mu_h\right)T_B.$$

Hence,

$$T_B(t) \geq T_B(0)e^{-(\tau+\delta_6+\mu_h)t} > 0. \tag{12}$$

For the Vaccinated class $\left(V_B\left(t\right)\right)$,from

$$\frac{dV_B}{dt} = \phi_2 S_h - \left(\phi_1 + \mu_h\right)V_B, \tag{13}$$

we obtain:

$$\frac{dV_B}{dt} \geq -\left(\phi_1 + \mu_h\right)V_B.$$

Therefore,

$$V_B(t) \geq V_B(0)e^{-(\phi_1+\mu_h)t} > 0. \tag{14}$$

For the Recovered class $\left(R_B\left(t\right)\right)$,from

$$\frac{dR_B}{dt} = \tau T_B - \left(\gamma + \mu_h\right)R_B. \tag{15}$$

we obtain:

$$\frac{dR_B}{dt} \geq -\left(\gamma + \mu_h\right)R_B.$$

Therefore,

$$R_B(t) \geq R_B(0)e^{-(\gamma+\mu_h)t} > 0. \tag{16}$$

Consequently,

$$(S_h(t), E_B(t), I_A(t), I_C(t), T_B(t), V_B(t), R_B(t)) \in R_+^7$$

for all $(t \geq 0)$. Hence, all solutions of the hepatitis B model remain non-negative for all time $(t \geq 0)$, proving that the model is epidemiologically and mathematically well posed in the feasible region.

Furthermore, the feasible region is given by:

$$\Omega = \left\{(S_h, E_B, I_A, I_C, T_B, V_B, R_B) \in R_+^7 : N_h = S_h + E_B + I_A + I_C + T_B + V_B + R_B \leq \frac{\Lambda_h}{\mu_h}\right\}. \tag{17}$$

Therefore, Ω is positively invariant and attracts all solutions of the system. Hence, the model is biologically meaningful and mathematically well posed in the region Ω.

Invariant Region of Hepatitis B

Theorem 2: if the initial value $S_h, E_B, I_A, I_C, T_B, V_B, R_B$, Then the solutions of hepatitis B model are non-negative.

Proof.

To prove the invariant region, we need to show that the solution of the model is visible.

The total population of the model is:

$$\frac{dN_h}{dt} = \Lambda_h - \mu_h S_h - \mu_h E_B - \mu_h I_A - \mu_h I_C$$

$$-\mu_h T_B - \mu_h V_B - \mu_h R_B - \delta_1 I_A - \delta_3 I_C - \delta_6 T_B,$$

$$\frac{dN_h}{dt} = \Lambda_h - \mu_h\left(S_h + E_B + I_A + I_C + T_B + V_B + R_B\right) - \delta_1 I_A - \delta_3 I_C - \delta_6 T_B,$$

$$\frac{dN_h}{dt} \leq \Lambda_h - \mu_h N_h,$$

$$\frac{dN_h}{dt} + \mu_h N_h \leq \Lambda_h,$$

$$N_h\left(t\right) \leq \left(\frac{\Lambda_h}{\mu_h} + N_h\left(0\right) - \frac{\Lambda_h}{\mu_h}\right)e^{-ut}. \tag{18}$$

This shows that the model has a unique solution and is invariant in the region $\Omega = \Omega_B$. where the solution is contained and the model is epidemiologically well posed and thereby meaningful.

Disease Free Equilibrium Point

Hepatitis B transmission model, the Disease-Free Equilibrium Point is the equilibrium state where **there is no Hepatitis B infection in the population**.

At Disease free equilibrium point

$$\left(S_h \neq 0, E_B = 0, I_A = 0, I_C = 0, T_B = 0, V_B \neq 0, R_B = 0\right).$$

$$\left(S_h, E_B, I_A, I_C, T_B, V_B, R_B\right) = \left(\frac{\Lambda_h\left(\phi_1 + \mu_h\right)}{\mu_h\left(\phi_2 + \phi_1 + \mu_h\right)}, 0, 0, 0, 0, \frac{\phi_2 \Lambda_h}{\mu_h\left(\phi_2 + \phi_1 + \mu_h\right)}, 0\right). \tag{19}$$

Basic Reproduction Number

The basic reproduction number, R_0^B, is the number of new infections caused by one infected individual in a susceptible population. The Basic Reproduction number is determined by next generation method $R_0^B = \rho_B F_B V_B^{-1}$ as F_B is a non-negative matrix with other transition terms V_B and the largest Eigen value ρ_B.

$$F_B = \begin{pmatrix} 0 & \frac{\beta_1\left(\phi_1 + \mu_h\right)}{\phi_2 + \phi_1 + \mu_h} & \frac{\beta_2\left(\phi_1 + \mu_h\right)}{\phi_2 + \phi_1 + \mu_h} & \frac{\beta_3\left(\phi_1 + \mu_h\right)}{\phi_2 + \phi_1 + \mu_h} \\ 0 & \beta_b \Lambda_h & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$V_B = \begin{pmatrix} K_2 & 0 & 0 & 0 \\ -\theta_1 & K_3 & 0 & 0 \\ 0 & -\omega_1 & K_4 & 0 \\ 0 & -\varepsilon_6 & -\varepsilon_1 & K_5 \end{pmatrix}.$$

$$R_0^B = \frac{\begin{array}{c}\beta_b\Lambda_h K_2K_4K_5\mu_h + \beta_b\Lambda_h K_2K_4K_5\phi_1 + \beta_b\Lambda_h K_2K_4K_5\phi_2 + K_4K_5\beta_1\mu_h\theta_1 + K_4K_5\beta_1\phi_1\theta_1 \\ +K_4\beta_3\varepsilon_6\mu_h\theta_1 + K_4\beta_3\varepsilon_6\phi_1\theta_1 + K_5\beta_2\mu_h\omega_1\theta_1 + K_5\beta_2\omega_1\phi_1\theta_1 + \beta_3\varepsilon_1\mu_h\omega_1\theta_1 + \beta_3\varepsilon_1\omega_1\phi_1\theta_1\end{array}}{(\phi_2+\phi_1+\mu_h)K_2K_3K_4K_5}. \quad (20)$$

Where $K_1 = (\phi_2 + \mu_h)$, $K_2 = (\theta_1 + \mu_h)$,
$K_3 = (\varepsilon_6 + \omega_1 + \delta_1 + \mu_h)$, $K_4 = (\varepsilon_1 + \delta_3 + \mu_h)$,
$K_5 = (\tau + \delta_6 + \mu_h)$, $K_6 = (\phi_1 + \mu_h)$,
$K_7 = (\gamma + \mu_h)$.

The value of R_0^B is important for determining the transmission behavior of HBV. When $R_0^B < 1$, each infectious individual produces less than one new infection on average, indicating that HBV transmission will decline and the disease can eventually be controlled. When $R_0^B > 1$ each infectious individual produces more than one new infection on average, indicating that HBV transmission can persist within the population. The threshold case $R_0^B = 1$ represents the critical point separating disease elimination from persistence.

The expression for R_0^B also provides biological insight into factors that influence HBV transmission. Parameters that are related to increased infectious contact and transmission usually increase and therefore promote disease spread. Conversely, control parameters like vaccination and treatment decrease the transmission probability of HBV by decreasing the number of susceptible individuals and the infectiousness or period of infection, respectively. Therefore, the size of is a good measure to assess the effectiveness of the proposed vaccination and treatment strategies in controlling the transmission of Hepatitis B.

Endemic Equilibrium Point

The endemic equilibrium of the Hepatitis B model represents the long-term state in which Hepatitis B virus (HBV) persists within the human population. At the endemic equilibrium,

$(S_h \neq 0, E_B \neq 0, I_A \neq 0, I_C \neq 0, T_B \neq 0, V_B \neq 0, R_B \neq 0)$.

$$S_h^* = -\frac{K_2\left(K_3K_4K_5K_7 + \left(-\Lambda_h K_4K_5K_7\beta_b + \tau\gamma\left(K_4\varepsilon_6 + \omega_1\varepsilon_1\right)\right)\beta_b\right)K_6\Lambda_h}{K_2K_4K_5K_7\left(-K_1K_6 - K_6\lambda_b + \phi_1\phi_2\right)K_3 + \lambda_b\theta_1K_6\left(-\Lambda_h K_4K_5K_7\beta_b + \tau\gamma\left(K_4\varepsilon_6 + \omega_1\varepsilon_1\right)\right)},$$

$$E_B^* = -\frac{\lambda_b\left(K_3K_4K_5K_7 + \left(-\Lambda_h K_4K_5K_7\beta_b + \tau\gamma\left(K_4\varepsilon_6 + \omega_1\varepsilon_1\right)\right)\beta_b\right)K_6\Lambda_h}{\left(-K_2K_4K_5K_7K_3 + \theta_1\left(-\Lambda_h K_4K_5K_7\beta_b + \tau\gamma\left(K_4\varepsilon_6 + \omega_1\varepsilon_1\right)\right)\right)K_6\lambda_b + K_2K_4K_5K_7K_3\left(-K_1K_6 + \phi_1\phi_2\right)},$$

$$I_A^* = \frac{K_7K_5\left(\beta_b\left(-K_1K_6 - K_6\lambda_b + \phi_1\phi_2\right)K_2 - \lambda_b\theta_1K_6\right)K_4\Lambda_h}{K_2K_4K_5K_7\left(-K_1K_6 - K_6\lambda_b + \phi_1\phi_2\right)K_3 + \lambda_b\theta_1K_6\left(-\Lambda_h K_4K_5K_7\beta_b + \tau\gamma\left(K_4\varepsilon_6 + \omega_1\varepsilon_1\right)\right)},$$

$$I_C^* = \frac{K_7\omega_1K_5\left(\beta_b\left(-K_1K_6 - K_6\lambda_b + \phi_1\phi_2\right)K_2 - \lambda_b\theta_1K_6\right)\Lambda_h}{K_2K_4K_5K_7\left(-K_1K_6 - K_6\lambda_b + \phi_1\phi_2\right)K_3 + \lambda_b\theta_1K_6\left(-\Lambda_h K_4K_5K_7\beta_b + \tau\gamma\left(K_4\varepsilon_6 + \omega_1\varepsilon_1\right)\right)},$$

$$T_B^* = \frac{K_7\left(\beta_b\left(-K_1K_6 - K_6\lambda_b + \phi_1\phi_2\right)K_2 - \lambda_b\theta_1K_6\right)\left(K_4\varepsilon_6 + \omega_1\varepsilon_1\right)\Lambda_h}{K_2K_4K_5K_7\left(-K_1K_6 - K_6\lambda_b + \phi_1\phi_2\right)K_3 + \lambda_b\theta_1K_6\left(-\Lambda_h K_4K_5K_7\beta_b + \tau\gamma\left(K_4\varepsilon_6 + \omega_1\varepsilon_1\right)\right)}, \quad (21)$$

$$V_B^* = -\frac{\phi_2K_2\left(K_3K_4K_5K_7 + \left(\left(-K_5K_7\Lambda_h\beta_b + \gamma\tau\varepsilon_6\right)K_4 + \tau\omega_1\varepsilon_1\gamma\right)\beta_b\right)\Lambda_h}{\left(-K_3K_4K_5K_7\left(\lambda_b + K_1\right)K_2 + \lambda_b\theta_1\left(\left(-K_5K_7\Lambda_h\beta_b + \gamma\tau\varepsilon_6\right)K_4 + \tau\omega_1\varepsilon_1\gamma\right)\right)K_6 + K_2K_3K_4K_5K_7\phi_1\phi_2},$$

$$R_B^* = -\frac{\begin{pmatrix}K_1K_2K_4K_6\beta_b\varepsilon_6 + K_1K_2K_6\beta_b\varepsilon_1\omega_1 + K_2K_4K_6\beta_b\varepsilon_6\lambda_b - K_2K_4\beta_b\varepsilon_6\phi_1\phi_2 \\ +K_2K_6\beta_b\varepsilon_1\lambda_b\omega_1 - K_2\beta_b\varepsilon_1\omega_1\phi_1\phi_2 + K_4K_6\varepsilon_6\lambda_b\theta_1 + K_6\varepsilon_1\lambda_b\omega_1\theta_1\end{pmatrix}\tau\Lambda_h}{\begin{array}{c}-K_4K_5K_6K_7\Lambda_h\beta_b\lambda_b\theta_1 + \gamma\tau K_4K_6\varepsilon_6\lambda_b\theta_1 + \gamma\tau K_6\varepsilon_1\lambda_b\omega_1\theta_1 \\ -K_1K_2K_3K_4K_5K_6K_7 - K_2K_3K_4K_5K_6K_7\lambda_b + K_2K_3K_4K_5K_7\phi_1\phi_2\end{array}}.$$

Substituting into the force of infection: $\lambda_b = \dfrac{\left(\beta_1 I_A + \beta_2 I_C + \beta_3 T_B\right)}{N_h}$.

We have:

$$H_1 \lambda_b + H_2 = 0. \quad (22)$$

Where

$$H_1 = \begin{pmatrix} -K_4K_5K_6K_7\Lambda_h\beta_b^{\,2} + \gamma\tau K_4K_6\beta_b\varepsilon_6 + \gamma\tau K_6\beta_b\varepsilon_1\omega_1 + \tau K_2K_4K_6\beta_b\varepsilon_6 \\ +\tau K_2K_6\beta_b\varepsilon_1\omega_1 + K_2K_4K_5K_6K_7\beta_b + K_2K_4K_6K_7\beta_b\varepsilon_6 + K_2K_5K_6K_7\beta_b\omega_1 \\ +K_2K_6K_7\beta_b\varepsilon_1\omega_1 + \tau K_4K_6\varepsilon_6\theta_1 + \tau K_6\varepsilon_1\omega_1\theta_1 + K_3K_4K_5K_6K_7 + K_4K_5K_6K_7\theta_1 \\ +K_4K_6K_7\varepsilon_6\theta_1 + K_5K_6K_7\omega_1\theta_1 + K_6K_7\varepsilon_1\omega_1\theta_1 \end{pmatrix},$$

$$H_2 = \begin{pmatrix} +\gamma\tau K_2K_4K_6\beta_b\varepsilon_6 + \gamma\tau K_2K_4\beta_b\varepsilon_6\phi_2 + \gamma\tau K_2K_6\beta_b\varepsilon_1\omega_1 + \gamma\tau K_2\beta_b\varepsilon_1\omega_1\phi_2 \\ +\tau K_1K_2K_4K_6\beta_b\varepsilon_6 + \tau K_1K_2K_6\beta_b\varepsilon_1\omega_1 - \tau K_2K_4\beta_b\varepsilon_6\phi_1\phi_2 - \tau K_2\beta_b\varepsilon_1\omega_1\phi_1\phi_2 \\ +K_1K_2K_4K_5K_6K_7\beta_b + K_1K_2K_4K_6K_7\beta_b\varepsilon_6 + K_1K_2K_5K_6K_7\beta_b\omega_1 + K_1K_2K_6K_7\beta_b\varepsilon_1\omega_1 \\ K_7\left(1 - \dfrac{\begin{matrix}\beta_b\Lambda_hK_2K_4K_5\mu_h + \beta_b\Lambda_hK_2K_4K_5\phi_1 + \beta_b\Lambda_hK_2K_4K_5\phi_2 + K_4K_5\beta_1\mu_h\theta_1 + K_4K_5\beta_1\phi_1\theta_1 \\ +K_4\beta_3\varepsilon_6\mu_h\theta_1 + K_4\beta_3\varepsilon_6\phi_1\theta_1 + K_5\beta_2\mu_h\omega_1\theta_1 + K_5\beta_2\omega_1\phi_1\theta_1 + \beta_3\varepsilon_1\mu_h\omega_1\theta_1 + \beta_3\varepsilon_1\omega_1\phi_1\theta_1\end{matrix}}{\left(\phi_2 + \phi_1 + \mu_h\right)K_2K_3K_4K_5}\right) \end{pmatrix},$$

$$H_2 = \begin{pmatrix} +\gamma\tau K_2K_4K_6\beta_b\varepsilon_6 + \gamma\tau K_2K_4\beta_b\varepsilon_6\phi_2 + \gamma\tau K_2K_6\beta_b\varepsilon_1\omega_1 + \gamma\tau K_2\beta_b\varepsilon_1\omega_1\phi_2 \\ +\tau K_1K_2K_4K_6\beta_b\varepsilon_6 + \tau K_1K_2K_6\beta_b\varepsilon_1\omega_1 - \tau K_2K_4\beta_b\varepsilon_6\phi_1\phi_2 - \tau K_2\beta_b\varepsilon_1\omega_1\phi_1\phi_2 \\ +K_1K_2K_4K_5K_6K_7\beta_b + K_1K_2K_4K_6K_7\beta_b\varepsilon_6 + K_1K_2K_5K_6K_7\beta_b\omega_1 + K_1K_2K_6K_7\beta_b\varepsilon_1\omega_1 \\ K_7\left(1 - R_0^B\right) \end{pmatrix}. \quad (23)$$

This means that the endemic equilibrium point of model (1) is stable if the rate of inflow of individuals into the compartments is matched by the rate of outflow due to infection, disease progression, treatment, recovery, vaccination and death. Specifically, if the value of the infected compartments is positive, the HBV virus is continuing to be sustained in the population. There are close connections between the existence and behavior of endemic equilibrium and the basic reproduction number R_0^B. When $R_0^B > 1$ HBV has the potential to persist in the population and an endemic equilibrium may exist. On the other hand, if $R_0^B < 1$, then transmission of HBV is likely to be reduced, in favor of elimination. The endemic equilibrium thus serves as an important indicator of the persistence of HBV in the long-term and the effectiveness of vaccination and treatment interventions.

Local Asymptotic stability at Disease free equilibrium point

Theorem 3: *The disease-free equilibrium point of Hepatitis Bsub-model is locally asymptotically stable (LAS) if* $R_0^B < 1$ *and unstable if* $R_0^B > 1$.

Proof.

The Jacobian matrix of the Hepatitis B virus model (1) evaluates at the disease-free equilibrium point ε_{0B} is given as:

$$J\left(\varepsilon_{0B}\right) = \begin{pmatrix} -K_1 & 0 & \dfrac{-\left(\beta_b\Lambda_h + \beta_1\left(\phi_1 + \mu_h\right)\right)}{\phi_2 + \phi_1 + \mu_h} & \dfrac{-\left(\beta_b\Lambda_h + \beta_2\left(\phi_1 + \mu_h\right)\right)}{\phi_2 + \phi_1 + \mu_h} & \dfrac{-\left(\beta_b\Lambda_h + \beta_3\left(\phi_1 + \mu_h\right)\right)}{\phi_2 + \phi_1 + \mu_h} & \phi_1 & \gamma \\ 0 & -K_2 & \dfrac{\left(\beta_b\Lambda_h + \beta_1\left(\phi_1 + \mu_h\right)\right)}{\phi_2 + \phi_1 + \mu_h} & \dfrac{\left(\beta_b\Lambda_h + \beta_2\left(\phi_1 + \mu_h\right)\right)}{\phi_2 + \phi_1 + \mu_h} & \dfrac{\left(\beta_b\Lambda_h + \beta_3\left(\phi_1 + \mu_h\right)\right)}{\phi_2 + \phi_1 + \mu_h} & 0 & 0 \\ 0 & \theta_1 & \beta_b\Lambda_h - K_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega_1 & -K_4 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon_6 & \varepsilon_1 & -K_5 & 0 & 0 \\ \phi_2 & 0 & 0 & 0 & 0 & -K_6 & 0 \\ 0 & 0 & 0 & 0 & \tau & 0 & -K_7 \end{pmatrix},$$

Taking the characteristic polynomial of the above Jacobian matrix we have the polynomial below:

$$P\left(\lambda_B^*\right) = \lambda^7 + M_1\lambda^6 + M_2\lambda^5 + M_3\lambda^4 + M_4\lambda^3 + M_5\lambda^2 + M_6\lambda + M_7. \tag{24}$$

Where

$$M_1 = -\Lambda_h\beta_b + K_1 + K_2 + K_3 + K_4 + K_5 + K_6 + K_7,$$

$$M_2 = \frac{1}{\phi_2+\phi_1+\mu_h}\begin{pmatrix}
K_1\Lambda_h\beta_b\mu_h + K_1\Lambda_h\beta_b\phi_1 + K_1\Lambda_h\beta_b\phi_2 + K_2\Lambda_h\beta_b\mu_h + K_2\Lambda_h\beta_b\phi_1 \\
+K_2\Lambda_h\beta_b\phi_2 + K_4\Lambda_h\beta_b\mu_h + K_4\Lambda_h\beta_b\phi_1 + K_4\Lambda_h\beta_b\phi_2 + K_5\Lambda_h\beta_b\mu_h \\
+K_5\Lambda_h\beta_b\phi_1 + K_5\Lambda_h\beta_b\phi_2 + K_6\Lambda_h\beta_b\mu_h + K_6\Lambda_h\beta_b\phi_1 + K_6\Lambda_h\beta_b\phi_2 \\
+K_7\Lambda_h\beta_b\mu_h + K_7\Lambda_h\beta_b\phi_1 + K_7\Lambda_h\beta_b\phi_2 - K_1K_2\mu_h - K_1K_2\phi_1 \\
-K_1K_2\phi_2 - K_1K_3\mu_h - K_1K_3\phi_1 - K_1K_3\phi_2 - K_1K_4\mu_h - K_1K_4\phi_1 \\
-K_1K_4\phi_2 - K_1K_5\mu_h - K_1K_5\phi_1 - K_1K_5\phi_2 - K_1K_6\mu_h - K_1K_6\phi_1 \\
-K_1K_6\phi_2 - K_1K_7\mu_h - K_1K_7\phi_1 - K_1K_7\phi_2 - K_2K_3\mu_h - K_2K_3\phi_1 \\
-K_2K_3\phi_2 - K_2K_4\mu_h - K_2K_4\phi_1 - K_2K_4\phi_2 - K_2K_5\mu_h - K_2K_5\phi_1 \\
-K_2K_5\phi_2 - K_2K_6\mu_h - K_2K_6\phi_1 - K_2K_6\phi_2 - K_2K_7\mu_h - K_2K_7\phi_1 \\
-K_2K_7\phi_2 - K_3K_4\mu_h - K_3K_4\phi_1 - K_3K_4\phi_2 - K_3K_5\mu_h - K_3K_5\phi_1 \\
-K_3K_5\phi_2 - K_3K_6\mu_h - K_3K_6\phi_1 - K_3K_6\phi_2 - K_3K_7\mu_h - K_3K_7\phi_1 \\
-K_3K_7\phi_2 - K_4K_5\mu_h - K_4K_5\phi_1 - K_4K_5\phi_2 - K_4K_6\mu_h - K_4K_6\phi_1 \\
-K_4K_6\phi_2 - K_4K_7\mu_h - K_4K_7\phi_1 - K_4K_7\phi_2 - K_5K_6\mu_h - K_5K_6\phi_1 \\
-K_5K_6\phi_2 - K_5K_7\mu_h - K_5K_7\phi_1 - K_5K_7\phi_2 - K_6K_7\mu_h - K_6K_7\phi_1 \\
-K_6K_7\phi_2 + \Lambda_h\beta_b\theta_1 + \beta_1\mu_h\theta_1 + \beta_1\phi_1\theta_1 + \mu_h\phi_1\phi_2 + \phi_1^2\phi_2 + \phi_1\phi_2^2
\end{pmatrix},$$

$$M_3 = \frac{1}{\phi_2+\phi_1+\mu_h}\begin{pmatrix}
K_1K_2\Lambda_h\beta_b\mu_h + K_1K_2\Lambda_h\beta_b\phi_1 + K_1K_2\Lambda_h\beta_b\phi_2 + K_1K_4\Lambda_h\beta_b\mu_h + K_1K_4\Lambda_h\beta_b\phi_1 \\
+K_1K_4\Lambda_h\beta_b\phi_2 + K_1K_5\Lambda_h\beta_b\mu_h + K_1K_5\Lambda_h\beta_b\phi_1 + K_1K_5\Lambda_h\beta_b\phi_2 + K_1K_6\Lambda_h\beta_b\mu_h \\
+K_1K_6\Lambda_h\beta_b\phi_1 + K_1K_6\Lambda_h\beta_b\phi_2 + K_1K_7\Lambda_h\beta_b\mu_h + K_1K_7\Lambda_h\beta_b\phi_1 + K_1K_7\Lambda_h\beta_b\phi_2 \\
+K_2K_4\Lambda_h\beta_b\mu_h + K_2K_4\Lambda_h\beta_b\phi_1 + K_2K_4\Lambda_h\beta_b\phi_2 + K_2K_5\Lambda_h\beta_b\mu_h + K_2K_5\Lambda_h\beta_b\phi_1 \\
+K_2K_5\Lambda_h\beta_b\phi_2 + K_2K_6\Lambda_h\beta_b\mu_h + K_2K_6\Lambda_h\beta_b\phi_1 + K_2K_6\Lambda_h\beta_b\phi_2 + K_2K_7\Lambda_h\beta_b\mu_h \\
+K_2K_7\Lambda_h\beta_b\phi_1 + K_2K_7\Lambda_h\beta_b\phi_2 + K_4K_5\Lambda_h\beta_b\mu_h + K_4K_5\Lambda_h\beta_b\phi_1 + K_4K_5\Lambda_h\beta_b\phi_2 \\
+K_4K_6\Lambda_h\beta_b\mu_h + K_4K_6\Lambda_h\beta_b\phi_1 + K_4K_6\Lambda_h\beta_b\phi_2 + K_4K_7\Lambda_h\beta_b\mu_h + K_4K_7\Lambda_h\beta_b\phi_1 \\
+K_4K_7\Lambda_h\beta_b\phi_2 + K_5K_6\Lambda_h\beta_b\mu_h + K_5K_6\Lambda_h\beta_b\phi_1 + K_5K_6\Lambda_h\beta_b\phi_2 + K_5K_7\Lambda_h\beta_b\mu_h \\
+K_5K_7\Lambda_h\beta_b\phi_1 + K_5K_7\Lambda_h\beta_b\phi_2 + K_6K_7\Lambda_h\beta_b\mu_h + K_6K_7\Lambda_h\beta_b\phi_1 + K_6K_7\Lambda_h\beta_b\phi_2 \\
-\Lambda_h\beta_b\mu_h\phi_1\phi_2 - \Lambda_h\beta_b\phi_1^2\phi_2 - \Lambda_h\beta_b\phi_1\phi_2^2 - K_1K_2K_3\mu_h - K_1K_2K_3\phi_1 - K_1K_2K_3\phi_2 \\
-K_1K_2K_4\mu_h - K_1K_2K_4\phi_1 - K_1K_2K_4\phi_2 - K_1K_2K_5\mu_h - K_1K_2K_5\phi_1 - K_1K_2K_5\phi_2 \\
-K_1K_2K_6\mu_h - K_1K_2K_6\phi_1 - K_1K_2K_6\phi_2 - K_1K_2K_7\mu_h - K_1K_2K_7\phi_1 - K_1K_2K_7\phi_2 \\
-K_1K_3K_4\mu_h - K_1K_3K_4\phi_1 - K_1K_3K_4\phi_2 - K_1K_3K_5\mu_h - K_1K_3K_5\phi_1 - K_1K_3K_5\phi_2 \\
-K_1K_3K_6\mu_h - K_1K_3K_6\phi_1 - K_1K_3K_6\phi_2 - K_1K_3K_7\mu_h - K_1K_3K_7\phi_1 - K_1K_3K_7\phi_2 \\
-K_1K_4K_5\mu_h - K_1K_4K_5\phi_1 - K_1K_4K_5\phi_2 - K_1K_4K_6\mu_h - K_1K_4K_6\phi_1 - K_1K_4K_6\phi_2 \\
-K_1K_4K_7\mu_h - K_1K_4K_7\phi_1 - K_1K_4K_7\phi_2 - K_1K_5K_6\mu_h - K_1K_5K_6\phi_1 - K_1K_5K_6\phi_2 \\
-K_1K_5K_7\mu_h - K_1K_5K_7\phi_1 - K_1K_5K_7\phi_2 - K_1K_6K_7\mu_h - K_1K_6K_7\phi_1 - K_1K_6K_7\phi_2 \\
+K_1\Lambda_h\beta_b\theta_1 + K_1\beta_1\mu_h\theta_1 + K_1\beta_1\phi_1\theta_1 - K_2K_3K_4\mu_h - K_2K_3K_4\phi_1 - K_2K_3K_4\phi_2 \\
-K_2K_3K_5\mu_h - K_2K_3K_5\phi_1 - K_2K_3K_5\phi_2 - K_2K_3K_6\mu_h - K_2K_3K_6\phi_1 - K_2K_3K_6\phi_2 \\
-K_2K_3K_7\mu_h - K_2K_3K_7\phi_1 - K_2K_3K_7\phi_2 - K_2K_4K_5\mu_h - K_2K_4K_5\phi_1 - K_2K_4K_5\phi_2 \\
-K_2K_4K_6\mu_h - K_2K_4K_6\phi_1 - K_2K_4K_6\phi_2 - K_2K_4K_7\mu_h - K_2K_4K_7\phi_1 - K_2K_4K_7\phi_2 \\
-K_2K_5K_6\mu_h - K_2K_5K_6\phi_1 - K_2K_5K_6\phi_2 - K_2K_5K_7\mu_h - K_2K_5K_7\phi_1 - K_2K_5K_7\phi_2 \\
-K_2K_6K_7\mu_h - K_2K_6K_7\phi_1 - K_2K_6K_7\phi_2 + K_2\mu_h\phi_1\phi_2 + K_2\phi_1^2\phi_2 + K_2\phi_1\phi_2^2 \\
-K_3K_4K_5\mu_h - K_3K_4K_5\phi_1 - K_3K_4K_5\phi_2 - K_3K_4K_6\mu_h - K_3K_4K_6\phi_1 - K_3K_4K_6\phi_2 \\
-K_3K_4K_7\mu_h - K_3K_4K_7\phi_1 - K_3K_4K_7\phi_2 - K_3K_5K_6\mu_h - K_3K_5K_6\phi_1 - K_3K_5K_6\phi_2 \\
-K_3K_5K_7\mu_h - K_3K_5K_7\phi_1 - K_3K_5K_7\phi_2 - K_3K_6K_7\mu_h - K_3K_6K_7\phi_1 - K_3K_6K_7\phi_2 \\
+K_3\mu_h\phi_1\phi_2 + K_3\phi_1^2\phi_2 + K_3\phi_1\phi_2^2 - K_4K_5K_6\mu_h - K_4K_5K_6\phi_1 - K_4K_5K_6\phi_2 - K_4K_5K_7\mu_h \\
-K_4K_5K_7\phi_1 - K_4K_5K_7\phi_2 - K_4K_6K_7\mu_h - K_4K_6K_7\phi_1 - K_4K_6K_7\phi_2 + K_4\Lambda_h\beta_b\theta_1 \\
+K_4\beta_1\mu_h\theta_1 + K_4\beta_1\phi_1\theta_1 + K_4\mu_h\phi_1\phi_2 + K_4\phi_1^2\phi_2 + K_4\phi_1\phi_2^2 - K_5K_6K_7\mu_h - K_5K_6K_7\phi_1 \\
-K_5K_6K_7\phi_2 + K_5\Lambda_h\beta_b\theta_1 + K_5\beta_1\mu_h\theta_1 + K_5\beta_1\phi_1\theta_1 + K_5\mu_h\phi_1\phi_2 + K_5\phi_1^2\phi_2 + K_5\phi_1\phi_2^2 \\
+K_6\Lambda_h\beta_b\theta_1 + K_6\beta_1\mu_h\theta_1 + K_6\beta_1\phi_1\theta_1 + K_7\Lambda_h\beta_b\theta_1 + K_7\beta_1\mu_h\theta_1 + K_7\beta_1\phi_1\theta_1 + K_7\mu_h\phi_1\phi_2 \\
+K_7\phi_1^2\phi_2 + K_7\phi_1\phi_2^2 + \Lambda_h\beta_b\varepsilon_6\theta_1 + \Lambda_h\beta_b\omega_1\theta_1 + \beta_2\mu_h\omega_1\theta_1 + \beta_2\omega_1\phi_1\theta_1 + \beta_3\varepsilon_6\mu_h\theta_1 + \beta_3\varepsilon_6\phi_1\theta_1
\end{pmatrix},$$

$$
M_4 = \frac{1}{\phi_2+\phi_1+\mu_h}\left(\begin{array}{l}
K_1K_2K_4\Lambda_h\beta_b\mu_h + K_1K_2K_4\Lambda_h\beta_b\phi_1 + K_1K_2K_4\Lambda_h\beta_b\phi_2 + K_1K_2K_5\Lambda_h\beta_b\mu_h \\
+K_1K_2K_5\Lambda_h\beta_b\phi_1 + K_1K_2K_5\Lambda_h\beta_b\phi_2 + K_1K_2K_6\Lambda_h\beta_b\mu_h + K_1K_2K_6\Lambda_h\beta_b\phi_1 \\
+K_1K_2K_6\Lambda_h\beta_b\phi_2 + K_1K_2K_7\Lambda_h\beta_b\mu_h + K_1K_2K_7\Lambda_h\beta_b\phi_1 + K_1K_2K_7\Lambda_h\beta_b\phi_2 \\
+K_1K_4K_5\Lambda_h\beta_b\mu_h + K_1K_4K_5\Lambda_h\beta_b\phi_1 + K_1K_4K_5\Lambda_h\beta_b\phi_2 + K_1K_4K_6\Lambda_h\beta_b\mu_h \\
+K_1K_4K_6\Lambda_h\beta_b\phi_1 + K_1K_4K_6\Lambda_h\beta_b\phi_2 + K_1K_4K_7\Lambda_h\beta_b\mu_h + K_1K_4K_7\Lambda_h\beta_b\phi_1 \\
+K_1K_4K_7\Lambda_h\beta_b\phi_2 + K_1K_5K_6\Lambda_h\beta_b\mu_h + K_1K_5K_6\Lambda_h\beta_b\phi_1 + K_1K_5K_6\Lambda_h\beta_b\phi_2 \\
+K_1K_5K_7\Lambda_h\beta_b\mu_h + K_1K_5K_7\Lambda_h\beta_b\phi_1 + K_1K_5K_7\Lambda_h\beta_b\phi_2 + K_1K_6K_7\Lambda_h\beta_b\mu_h \\
+K_1K_6K_7\Lambda_h\beta_b\phi_1 + K_1K_6K_7\Lambda_h\beta_b\phi_2 + K_2K_4K_5\Lambda_h\beta_b\mu_h + K_2K_4K_5\Lambda_h\beta_b\phi_1 \\
+K_2K_4K_5\Lambda_h\beta_b\phi_2 + K_2K_4K_6\Lambda_h\beta_b\mu_h + K_2K_4K_6\Lambda_h\beta_b\phi_1 + K_2K_4K_6\Lambda_h\beta_b\phi_2 \\
+K_2K_4K_7\Lambda_h\beta_b\mu_h + K_2K_4K_7\Lambda_h\beta_b\phi_1 + K_2K_4K_7\Lambda_h\beta_b\phi_2 + K_2K_5K_6\Lambda_h\beta_b\mu_h \\
+K_2K_5K_6\Lambda_h\beta_b\phi_1 + K_2K_5K_6\Lambda_h\beta_b\phi_2 + K_2K_5K_7\Lambda_h\beta_b\mu_h + K_2K_5K_7\Lambda_h\beta_b\phi_1 \\
+K_2K_5K_7\Lambda_h\beta_b\phi_2 + K_2K_6K_7\Lambda_h\beta_b\mu_h + K_2K_6K_7\Lambda_h\beta_b\phi_1 + K_2K_6K_7\Lambda_h\beta_b\phi_2 \\
-K_2\Lambda_h\beta_b\mu_h\phi_1\phi_2 - K_2\Lambda_h\beta_b\phi_1^2\phi_2 - K_2\Lambda_h\beta_b\phi_1\phi_2^2 + K_4K_5K_6\Lambda_h\beta_b\mu_h \\
+K_4K_5K_6\Lambda_h\beta_b\phi_1 + K_4K_5K_6\Lambda_h\beta_b\phi_2 + K_4K_5K_7\Lambda_h\beta_b\mu_h + K_4K_5K_7\Lambda_h\beta_b\phi_1 \\
+K_4K_5K_7\Lambda_h\beta_b\phi_2 + K_4K_6K_7\Lambda_h\beta_b\mu_h + K_4K_6K_7\Lambda_h\beta_b\phi_1 + K_4K_6K_7\Lambda_h\beta_b\phi_2 \\
-K_4\Lambda_h\beta_b\mu_h\phi_1\phi_2 - K_4\Lambda_h\beta_b\phi_1^2\phi_2 - K_4\Lambda_h\beta_b\phi_1\phi_2^2 + K_5K_6K_7\Lambda_h\beta_b\mu_h \\
+K_5K_6K_7\Lambda_h\beta_b\phi_1 + K_5K_6K_7\Lambda_h\beta_b\phi_2 - K_5\Lambda_h\beta_b\mu_h\phi_1\phi_2 - K_5\Lambda_h\beta_b\phi_1^2\phi_2 \\
-K_5\Lambda_h\beta_b\phi_1\phi_2^2 - K_7\Lambda_h\beta_b\mu_h\phi_1\phi_2 - K_7\Lambda_h\beta_b\phi_1^2\phi_2 - K_7\Lambda_h\beta_b\phi_1\phi_2^2 - K_1K_2K_3K_4\mu_h \\
-K_1K_2K_3K_4\phi_1 - K_1K_2K_3K_4\phi_2 - K_1K_2K_3K_5\mu_h - K_1K_2K_3K_5\phi_1 - K_1K_2K_3K_5\phi_2 \\
-K_1K_2K_3K_6\mu_h - K_1K_2K_3K_6\phi_1 - K_1K_2K_3K_6\phi_2 - K_1K_2K_3K_7\mu_h - K_1K_2K_3K_7\phi_1 \\
-K_1K_2K_3K_7\phi_2 - K_1K_2K_4K_5\mu_h - K_1K_2K_4K_5\phi_1 - K_1K_2K_4K_5\phi_2 - K_1K_2K_4K_6\mu_h \\
-K_1K_2K_4K_6\phi_1 - K_1K_2K_4K_6\phi_2 - K_1K_2K_4K_7\mu_h - K_1K_2K_4K_7\phi_1 - K_1K_2K_4K_7\phi_2 \\
-K_1K_2K_5K_6\mu_h - K_1K_2K_5K_6\phi_1 - K_1K_2K_5K_6\phi_2 - K_1K_2K_5K_7\mu_h - K_1K_2K_5K_7\phi_1 \\
-K_1K_2K_5K_7\phi_2 - K_1K_2K_6K_7\mu_h - K_1K_2K_6K_7\phi_1 - K_1K_2K_6K_7\phi_2 - K_1K_3K_4K_5\mu_h \\
-K_1K_3K_4K_5\phi_1 - K_1K_3K_4K_5\phi_2 - K_1K_3K_4K_6\mu_h - K_1K_3K_4K_6\phi_1 - K_1K_3K_4K_6\phi_2 \\
-K_1K_3K_4K_7\mu_h - K_1K_3K_4K_7\phi_1 - K_1K_3K_4K_7\phi_2 - K_1K_3K_5K_6\mu_h - K_1K_3K_5K_6\phi_1 \\
-K_1K_3K_5K_6\phi_2 - K_1K_3K_5K_7\mu_h - K_1K_3K_5K_7\phi_1 - K_1K_3K_5K_7\phi_2 - K_1K_3K_6K_7\mu_h \\
-K_1K_3K_6K_7\phi_1 - K_1K_3K_6K_7\phi_2 - K_1K_4K_5K_6\mu_h - K_1K_4K_5K_6\phi_1 - K_1K_4K_5K_6\phi_2 \\
-K_1K_4K_5K_7\mu_h - K_1K_4K_5K_7\phi_1 - K_1K_4K_5K_7\phi_2 - K_1K_4K_6K_7\mu_h - K_1K_4K_6K_7\phi_1 \\
-K_1K_4K_6K_7\phi_2 + K_1K_4\Lambda_h\beta_b\theta_1 + K_1K_4\beta_1\mu_h\theta_1 + K_1K_4\beta_1\phi_1\theta_1 - K_1K_5K_6K_7\mu_h \\
-K_1K_5K_6K_7\phi_1 - K_1K_5K_6K_7\phi_2 + K_1K_5\Lambda_h\beta_b\theta_1 + K_1K_5\beta_1\mu_h\theta_1 + K_1K_5\beta_1\phi_1\theta_1 \\
+K_1K_6\Lambda_h\beta_b\theta_1 + K_1K_6\beta_1\mu_h\theta_1 + K_1K_6\beta_1\phi_1\theta_1 + K_1K_7\Lambda_h\beta_b\theta_1 + K_1K_7\beta_1\mu_h\theta_1 \\
+K_1K_7\beta_1\phi_1\theta_1 + K_1\Lambda_h\beta_b\epsilon_6\theta_1 + K_1\Lambda_h\beta_b\omega_1\theta_1 + K_1\beta_2\mu_h\omega_1\theta_1 + K_1\beta_2\omega_1\phi_1\theta_1 \\
+K_1\beta_3\varepsilon_6\mu_h\theta_1 + K_1\beta_3\epsilon_6\phi_1\theta_1 - K_2K_3K_4K_5\mu_h - K_2K_3K_4K_5\phi_1 - K_2K_3K_4K_5\phi_2 \\
-K_2K_3K_4K_6\mu_h - K_2K_3K_4K_6\phi_1 - K_2K_3K_4K_6\phi_2 - K_2K_3K_4K_7\mu_h - K_2K_3K_4K_7\phi_1 \\
-K_2K_3K_4K_7\phi_2 - K_2K_3K_5K_6\mu_h - K_2K_3K_5K_6\phi_1 - K_2K_3K_5K_6\phi_2 - K_2K_3K_5K_7\mu_h \\
-K_2K_3K_5K_7\phi_1 - K_2K_3K_5K_7\phi_2 - K_2K_3K_6K_7\mu_h - K_2K_3K_6K_7\phi_1 - K_2K_3K_6K_7\phi_2 \\
+K_2K_3\mu_h\phi_1\phi_2 + K_2K_3\phi_1^2\phi_2 + K_2K_3\phi_1\phi_2^2 - K_2K_4K_5K_6\mu_h - K_2K_4K_5K_6\phi_1 \\
-K_2K_4K_5K_6\phi_2 - K_2K_4K_5K_7\mu_h - K_2K_4K_5K_7\phi_1 - K_2K_4K_5K_7\phi_2 - K_2K_4K_6K_7\mu_h \\
-K_2K_4K_6K_7\phi_1 - K_2K_4K_6K_7\phi_2 + K_2K_4\mu_h\phi_1\phi_2 + K_2K_4\phi_1^2\phi_2 + K_2K_4\phi_1\phi_2^2 \\
-K_2K_5K_6K_7\mu_h - K_2K_5K_6K_7\phi_1 - K_2K_5K_6K_7\phi_2 + K_2K_5\mu_h\phi_1\phi_2 + K_2K_5\phi_1^2\phi_2 \\
+K_2K_5\phi_1\phi_2^2
\end{array}\right),
$$

$$
M_5 = \frac{1}{\phi_2+\phi_1+\mu_h}\left(\begin{array}{l}
K_1K_2K_4K_5\Lambda_h\beta_b\mu_h + K_1K_2K_4K_5\Lambda_h\beta_b\phi_1 + K_1K_2K_4K_5\Lambda_h\beta_b\phi_2 + K_1K_2K_4K_6\Lambda_h\beta_b\mu_h \\
+K_1K_2K_4K_6\Lambda_h\beta_b\phi_1 + K_1K_2K_4K_6\Lambda_h\beta_b\phi_2 + K_1K_2K_4K_7\Lambda_h\beta_b\mu_h + K_1K_2K_4K_7\Lambda_h\beta_b\phi_1 \\
+K_1K_2K_4K_7\Lambda_h\beta_b\phi_2 + K_1K_2K_5K_6\Lambda_h\beta_b\mu_h + K_1K_2K_5K_6\Lambda_h\beta_b\phi_1 + K_1K_2K_5K_6\Lambda_h\beta_b\phi_2 \\
+K_1K_2K_5K_7\Lambda_h\beta_b\mu_h + K_1K_2K_5K_7\Lambda_h\beta_b\phi_1 + K_1K_2K_5K_7\Lambda_h\beta_b\phi_2 + K_1K_2K_6K_7\Lambda_h\beta_b\mu_h \\
+K_1K_2K_6K_7\Lambda_h\beta_b\phi_1 + K_1K_2K_6K_7\Lambda_h\beta_b\phi_2 + K_1K_4K_5K_6\Lambda_h\beta_b\mu_h + K_1K_4K_5K_6\Lambda_h\beta_b\phi_1 \\
+K_1K_4K_5K_6\Lambda_h\beta_b\phi_2 + K_1K_4K_5K_7\Lambda_h\beta_b\mu_h + K_1K_4K_5K_7\Lambda_h\beta_b\phi_1 + K_1K_4K_5K_7\Lambda_h\beta_b\phi_2 \\
+K_1K_4K_6K_7\Lambda_h\beta_b\mu_h + K_1K_4K_6K_7\Lambda_h\beta_b\phi_1 + K_1K_4K_6K_7\Lambda_h\beta_b\phi_2 + K_1K_5K_6K_7\Lambda_h\beta_b\mu_h \\
+K_1K_5K_6K_7\Lambda_h\beta_b\phi_1 + K_1K_5K_6K_7\Lambda_h\beta_b\phi_2 + K_2K_4K_5K_6\Lambda_h\beta_b\mu_h + K_2K_4K_5K_6\Lambda_h\beta_b\phi_1 \\
+K_2K_4K_5K_6\Lambda_h\beta_b\phi_2 + K_2K_4K_5K_7\Lambda_h\beta_b\mu_h + K_2K_4K_5K_7\Lambda_h\beta_b\phi_1 + K_2K_4K_5K_7\Lambda_h\beta_b\phi_2 \\
+K_2K_4K_6K_7\Lambda_h\beta_b\mu_h + K_2K_4K_6K_7\Lambda_h\beta_b\phi_1 + K_2K_4K_6K_7\Lambda_h\beta_b\phi_2 - K_2K_4\Lambda_h\beta_b\mu_h\phi_1\phi_2 \\
-K_2K_4\Lambda_h\beta_b\phi_1{}^2\phi_2 - K_2K_4\Lambda_h\beta_b\phi_1\phi_2{}^2 + K_2K_5K_6K_7\Lambda_h\beta_b\mu_h + K_2K_5K_6K_7\Lambda_h\beta_b\phi_1 \\
+K_2K_5K_6K_7\Lambda_h\beta_b\phi_2 - K_2K_5\Lambda_h\beta_b\mu_h\phi_1\phi_2 - K_2K_5\Lambda_h\beta_b\phi_1{}^2\phi_2 - K_2K_5\Lambda_h\beta_b\phi_1\phi_2{}^2 \\
-K_2K_7\Lambda_h\beta_b\mu_h\phi_1\phi_2 - K_2K_7\Lambda_h\beta_b\phi_1{}^2\phi_2 - K_2K_7\Lambda_h\beta_b\phi_1\phi_2{}^2 + K_4K_5K_6K_7\Lambda_h\beta_b\mu_h \\
+K_4K_5K_6K_7\Lambda_h\beta_b\phi_1 + K_4K_5K_6K_7\Lambda_h\beta_b\phi_2 - K_4K_5\Lambda_h\beta_b\mu_h\phi_1\phi_2 - K_4K_5\Lambda_h\beta_b\phi_1{}^2\phi_2 \\
-K_4K_5\Lambda_h\beta_b\phi_1\phi_2{}^2 - K_4K_7\Lambda_h\beta_b\mu_h\phi_1\phi_2 - K_4K_7\Lambda_h\beta_b\phi_1{}^2\phi_2 - K_4K_7\Lambda_h\beta_b\phi_1\phi_2{}^2 \\
-K_5K_7\Lambda_h\beta_b\mu_h\phi_1\phi_2 - K_5K_7\Lambda_h\beta_b\phi_1{}^2\phi_2 - K_5K_7\Lambda_h\beta_b\phi_1\phi_2{}^2 - K_1K_2K_3K_4K_5\mu_h \\
-K_1K_2K_3K_4K_5\phi_1 - K_1K_2K_3K_4K_5\phi_2 - K_1K_2K_3K_4K_6\mu_h - K_1K_2K_3K_4K_6\phi_1 - K_1K_2K_3K_4K_6\phi_2 \\
-K_1K_2K_3K_4K_7\mu_h - K_1K_2K_3K_4K_7\phi_1 - K_1K_2K_3K_4K_7\phi_2 - K_1K_2K_3K_5K_6\mu_h - K_1K_2K_3K_5K_6\phi_1 \\
-K_1K_2K_3K_5K_6\phi_2 - K_1K_2K_3K_5K_7\mu_h - K_1K_2K_3K_5K_7\phi_1 - K_1K_2K_3K_5K_7\phi_2 - K_1K_2K_3K_6K_7\mu_h \\
-K_1K_2K_3K_6K_7\phi_1 - K_1K_2K_3K_6K_7\phi_2 - K_1K_2K_4K_5K_6\mu_h - K_1K_2K_4K_5K_6\phi_1 - K_1K_2K_4K_5K_6\phi_2 \\
-K_1K_2K_4K_5K_7\mu_h - K_1K_2K_4K_5K_7\phi_1 - K_1K_2K_4K_5K_7\phi_2 - K_1K_2K_4K_6K_7\mu_h - K_1K_2K_4K_6K_7\phi_1 \\
-K_1K_2K_4K_6K_7\phi_2 - K_1K_2K_5K_6K_7\mu_h - K_1K_2K_5K_6K_7\phi_1 - K_1K_2K_5K_6K_7\phi_2 - K_1K_3K_4K_5K_6\mu_h \\
-K_1K_3K_4K_5K_6\phi_1 - K_1K_3K_4K_5K_6\phi_2 - K_1K_3K_4K_5K_7\mu_h - K_1K_3K_4K_5K_7\phi_1 - K_1K_3K_4K_5K_7\phi_2 \\
-K_1K_3K_4K_6K_7\mu_h - K_1K_3K_4K_6K_7\phi_1 - K_1K_3K_4K_6K_7\phi_2 - K_1K_3K_5K_6K_7\mu_h - K_1K_3K_5K_6K_7\phi_1 \\
-K_1K_3K_5K_6K_7\phi_2 - K_1K_4K_5K_6K_7\mu_h - K_1K_4K_5K_6K_7\phi_1 - K_1K_4K_5K_6K_7\phi_2 + K_1K_4K_5\Lambda_h\beta_b\theta_1 \\
+K_1K_4K_5\beta_1\mu_h\theta_1 + K_1K_4K_5\beta_1\phi_1\theta_1 + K_1K_4K_6\Lambda_h\beta_b\theta_1 + K_1K_4K_6\beta_1\mu_h\theta_1 + K_1K_4K_6\beta_1\phi_1\theta_1 \\
+K_1K_4K_7\Lambda_h\beta_b\theta_1 + K_1K_4K_7\beta_1\mu_h\theta_1 + K_1K_4K_7\beta_1\phi_1\theta_1 + K_1K_4\Lambda_h\beta_b\epsilon_6\theta_1 + K_1K_4\beta_3\epsilon_6\mu_h\theta_1 \\
+K_1K_4\beta_3\epsilon_6\phi_1\theta_1 + K_1K_5K_6\Lambda_h\beta_b\theta_1 + K_1K_5K_6\beta_1\mu_h\theta_1 + K_1K_5K_6\beta_1\phi_1\theta_1 + K_1K_5K_7\Lambda_h\beta_b\theta_1 \\
+K_1K_5K_7\beta_1\mu_h\theta_1 + K_1K_5K_7\beta_1\phi_1\theta_1 + K_1K_5\Lambda_h\beta_b\omega_1\theta_1 + K_1K_5\beta_2\mu_h\omega_1\theta_1 + K_1K_5\beta_2\omega_1\phi_1\theta_1 \\
+K_1K_6K_7\Lambda_h\beta_b\theta_1 + K_1K_6K_7\beta_1\mu_h\theta_1 + K_1K_6K_7\beta_1\phi_1\theta_1 + K_1K_6\Lambda_h\beta_b\epsilon_6\theta_1 + K_1K_6\Lambda_h\beta_b\omega_1\theta_1 \\
+K_1K_6\beta_2\mu_h\omega_1\theta_1 + K_1K_6\beta_2\omega_1\phi_1\theta_1 + K_1K_6\beta_3\epsilon_6\mu_h\theta_1 + K_1K_6\beta_3\epsilon_6\phi_1\theta_1 + K_1K_7\Lambda_h\beta_b\epsilon_6\theta_1 \\
+K_1K_7\Lambda_h\beta_b\omega_1\theta_1 + K_1K_7\beta_2\mu_h\omega_1\theta_1 + K_1K_7\beta_2\omega_1\phi_1\theta_1 + K_1K_7\beta_3\epsilon_6\mu_h\theta_1 + K_1K_7\beta_3\epsilon_6\phi_1\theta_1 \\
+K_1\Lambda_h\beta_b\epsilon_1\omega_1\theta_1 + K_1\beta_3\epsilon_1\mu_h\omega_1\theta_1 + K_1\beta_3\epsilon_1\omega_1\phi_1\theta_1 - K_2K_3K_4K_5K_6\mu_h - K_2K_3K_4K_5K_6\phi_1 \\
-K_2K_3K_4K_5K_6\phi_2 - K_2K_3K_4K_5K_7\mu_h - K_2K_3K_4K_5K_7\phi_1 + K_2K_3K_4\phi_1\phi_2{}^2 - K_2K_3K_5K_6K_7\mu_h \\
-K_2K_3K_5K_6K_7\phi_1 - K_2K_3K_5K_6K_7\phi_2 + K_2K_3K_5\mu_h\phi_1\phi_2 + K_2K_3K_5\phi_1{}^2\phi_2 + K_2K_3K_5\phi_1\phi_2{}^2 \\
+K_2K_3K_7\mu_h\phi_1\phi_2 + K_2K_3K_7\phi_1{}^2\phi_2 + K_2K_3K_7\phi_1\phi_2{}^2 - K_2K_4K_5K_6K_7\mu_h - K_2K_4K_5K_6K_7\phi_1 \\
-K_2K_4K_5K_6K_7\phi_2
\end{array}\right),
$$

$$
M_6 = \frac{1}{\phi_2+\phi_1+\mu_h}\left(\begin{array}{l}
K_1K_2K_4K_5K_6\Lambda_h\beta_b\mu_h + K_1K_2K_4K_5K_6\Lambda_h\beta_b\phi_1 + K_1K_2K_4K_5K_6\Lambda_h\beta_b\phi_2 \\
+K_1K_2K_4K_5K_7\Lambda_h\beta_b\mu_h + K_1K_2K_4K_5K_7\Lambda_h\beta_b\phi_1 + K_1K_2K_4K_5K_7\Lambda_h\beta_b\phi_2 \\
+K_1K_2K_4K_6K_7\Lambda_h\beta_b\mu_h + K_1K_2K_4K_6K_7\Lambda_h\beta_b\phi_1 + K_1K_2K_4K_6K_7\Lambda_h\beta_b\phi_2 \\
+K_1K_2K_5K_6K_7\Lambda_h\beta_b\mu_h + K_1K_2K_5K_6K_7\Lambda_h\beta_b\phi_1 + K_1K_2K_5K_6K_7\Lambda_h\beta_b\phi_2 \\
+K_1K_4K_5K_6K_7\Lambda_h\beta_b\mu_h + K_1K_4K_5K_6K_7\Lambda_h\beta_b\phi_1 + K_1K_4K_5K_6K_7\Lambda_h\beta_b\phi_2 \\
+K_2K_4K_5K_6K_7\Lambda_h\beta_b\mu_h + K_2K_4K_5K_6K_7\Lambda_h\beta_b\phi_1 + K_2K_4K_5K_6K_7\Lambda_h\beta_b\phi_2 \\
-K_2K_4K_5\Lambda_h\beta_b\mu_h\phi_1\phi_2 - K_2K_4K_5\Lambda_h\beta_b\phi_1^2\phi_2 - K_2K_4K_5\Lambda_h\beta_b\phi_1\phi_2^2 \\
-K_2K_4K_7\Lambda_h\beta_b\mu_h\phi_1\phi_2 - K_2K_4K_7\Lambda_h\beta_b\phi_1^2\phi_2 - K_2K_4K_7\Lambda_h\beta_b\phi_1\phi_2^2 \\
-K_2K_5K_7\Lambda_h\beta_b\mu_h\phi_1\phi_2 - K_2K_5K_7\Lambda_h\beta_b\phi_1^2\phi_2 - K_2K_5K_7\Lambda_h\beta_b\phi_1\phi_2^2 \\
-K_4K_5K_7\Lambda_h\beta_b\mu_h\phi_1\phi_2 - K_4K_5K_7\Lambda_h\beta_b\phi_1^2\phi_2 - K_4K_5K_7\Lambda_h\beta_b\phi_1\phi_2^2 \\
-K_1K_2K_3K_4K_5K_6\mu_h - K_1K_2K_3K_4K_5K_6\phi_1 - K_1K_2K_3K_4K_5K_6\phi_2 \\
-K_1K_2K_3K_4K_5K_7\mu_h - K_1K_2K_3K_4K_5K_7\phi_1 - K_1K_2K_3K_4K_5K_7\phi_2 \\
-K_1K_2K_3K_4K_6K_7\mu_h - K_1K_2K_3K_4K_6K_7\phi_1 - K_1K_2K_3K_4K_6K_7\phi_2 \\
-K_1K_2K_3K_5K_6K_7\mu_h - K_1K_2K_3K_5K_6K_7\phi_1 - K_1K_2K_3K_5K_6K_7\phi_2 \\
-K_1K_2K_4K_5K_6K_7\mu_h - K_1K_2K_4K_5K_6K_7\phi_1 - K_1K_2K_4K_5K_6K_7\phi_2 \\
-K_1K_3K_4K_5K_6K_7\mu_h - K_1K_3K_4K_5K_6K_7\phi_1 - K_1K_3K_4K_5K_6K_7\phi_2 \\
+K_1K_4K_5K_6\Lambda_h\beta_b\theta_1 + K_1K_4K_5K_6\beta_1\mu_h\theta_1 + K_1K_4K_5K_6\beta_1\phi_1\theta_1 \\
+K_1K_4K_5K_7\Lambda_h\beta_b\theta_1 + K_1K_4K_5K_7\beta_1\mu_h\theta_1 + K_1K_4K_5K_7\beta_1\phi_1\theta_1 \\
+K_1K_4K_6K_7\Lambda_h\beta_b\theta_1 + K_1K_4K_6K_7\beta_1\mu_h\theta_1 + K_1K_4K_6K_7\beta_1\phi_1\theta_1 \\
+K_1K_4K_6\Lambda_h\beta_b\varepsilon_6\theta_1 + K_1K_4K_6\beta_3\varepsilon_6\mu_h\theta_1 + K_1K_4K_6\beta_3\varepsilon_6\phi_1\theta_1 \\
+K_1K_4K_7\Lambda_h\beta_b\epsilon_6\theta_1 + K_1K_4K_7\beta_3\epsilon_6\mu_h\theta_1 + K_1K_4K_7\beta_3\epsilon_6\phi_1\theta_1 \\
+K_1K_5K_6K_7\Lambda_h\beta_b\theta_1 + K_1K_5K_6K_7\beta_1\mu_h\theta_1 + K_1K_5K_6K_7\beta_1\phi_1\theta_1 \\
+K_1K_5K_6\Lambda_h\beta_b\omega_1\theta_1 + K_1K_5K_6\beta_2\mu_h\omega_1\theta_1 + K_1K_5K_6\beta_2\omega_1\phi_1\theta_1 \\
+K_1K_5K_7\Lambda_h\beta_b\omega_1\theta_1 + K_1K_5K_7\beta_2\mu_h\omega_1\theta_1 + K_1K_5K_7\beta_2\omega_1\phi_1\theta_1 \\
+K_1K_6K_7\Lambda_h\beta_b\varepsilon_6\theta_1 + K_1K_6K_7\Lambda_h\beta_b\omega_1\theta_1 + K_1K_6K_7\beta_2\mu_h\omega_1\theta_1 \\
+K_1K_6K_7\beta_2\omega_1\phi_1\theta_1 + K_1K_6K_7\beta_3\varepsilon_6\mu_h\theta_1 + K_1K_6K_7\beta_3\varepsilon_6\phi_1\theta_1 \\
+K_1K_6\Lambda_h\beta_b\varepsilon_1\omega_1\theta_1 + K_1K_6\beta_3\varepsilon_1\mu_h\omega_1\theta_1 + K_1K_6\beta_3\varepsilon_1\omega_1\phi_1\theta_1 \\
+K_1K_7\Lambda_h\beta_b\varepsilon_1\omega_1\theta_1 + K_1K_7\beta_3\varepsilon_1\mu_h\omega_1\theta_1 + K_1K_7\beta_3\varepsilon_1\omega_1\phi_1\theta_1 \\
-K_2K_3K_4K_5K_6K_7\mu_h - K_2K_3K_4K_5K_6K_7\phi_1 - K_2K_3K_4K_5K_6K_7\phi_2 \\
+K_2K_3K_4K_5\mu_h\phi_1\phi_2 + K_2K_3K_4K_5\phi_1^2\phi_2 + K_2K_3K_4K_5\phi_1\phi_2^2 + K_2K_3K_4K_7\mu_h\phi_1\phi_2 \\
+K_2K_3K_4K_7\phi_1^2\phi_2 + K_2K_3K_4K_7\phi_1\phi_2^2 + K_2K_3K_5K_7\mu_h\phi_1\phi_2 + K_2K_3K_5K_7\phi_1^2\phi_2 \\
+K_2K_3K_5K_7\phi_1\phi_2^2 + K_2K_4K_5K_7\mu_h\phi_1\phi_2 + K_2K_4K_5K_7\phi_1^2\phi_2 + K_2K_4K_5K_7\phi_1\phi_2^2 \\
+K_3K_4K_5K_7\mu_h\phi_1\phi_2 + K_3K_4K_5K_7\phi_1^2\phi_2 + K_3K_4K_5K_7\phi_1\phi_2^2 + K_4K_5K_6K_7\Lambda_h\beta_b\theta_1 \\
+K_4K_5K_6K_7\beta_1\mu_h\theta_1 + K_4K_5K_6K_7\beta_1\phi_1\theta_1 - K_4K_5\Lambda_h\beta_b\phi_1\phi_2\theta_1 - K_4K_5\beta_1\mu_h\phi_1\phi_2\theta_1 \\
-K_4K_5\beta_1\phi_1^2\phi_2\theta_1 + K_4K_6K_7\Lambda_h\beta_b\varepsilon_6\theta_1 + K_4K_6K_7\beta_3\varepsilon_6\mu_h\theta_1 + K_4K_6K_7\beta_3\varepsilon_6\phi_1\theta_1 \\
-K_4K_7\Lambda_h\beta_b\phi_1\phi_2\theta_1 - K_4K_7\beta_1\mu_h\phi_1\phi_2\theta_1 - K_4K_7\beta_1\phi_1^2\phi_2\theta_1 - K_4\Lambda_h\beta_b\varepsilon_6\phi_1\phi_2\theta_1 \\
-K_4\beta_3\epsilon_6\mu_h\phi_1\phi_2\theta_1 - K_4\beta_3\varepsilon_6\phi_1^2\phi_2\theta_1 + K_5K_6K_7\Lambda_h\beta_b\omega_1\theta_1 + K_5K_6K_7\beta_2\mu_h\omega_1\theta_1 \\
+K_5K_6K_7\beta_2\omega_1\phi_1\theta_1 - K_5K_7\Lambda_h\beta_b\phi_1\phi_2\theta_1 - K_5K_7\beta_1\mu_h\phi_1\phi_2\theta_1 - K_5K_7\beta_1\phi_1^2\phi_2\theta_1 \\
-K_5\Lambda_h\beta_b\omega_1\phi_1\phi_2\theta_1 - K_5\beta_2\mu_h\omega_1\phi_1\phi_2\theta_1 - K_5\beta_2\omega_1\phi_1^2\phi_2\theta_1 + K_6K_7\Lambda_h\beta_b\varepsilon_1\omega_1\theta_1
\end{array}\right),
$$

$$M_7 = \left(K_7 \left(\begin{array}{l} K_2K_4K_5\Lambda_h\beta_b\mu_h\phi_1\phi_2 + K_2K_4K_5\Lambda_h\beta_b\phi_1^{\,2}\phi_2 + K_2K_4K_5\Lambda_h\beta_b\phi_1\phi_2^{\,2} + K_1K_2K_3K_4K_5K_6\mu_h \\ +K_1K_2K_3K_4K_5K_6\phi_1 + K_1K_2K_3K_4K_5K_6\phi_2 + K_4K_5\Lambda_h\beta_b\phi_1\phi_2\theta_1 + K_4K_5\beta_1\mu_h\phi_1\phi_2\theta_1 \\ +K_4K_5\beta_1\phi_1^{\,2}\phi_2\theta_1 + K_4\Lambda_h\beta_b\epsilon_6\phi_1\phi_2\theta_1 + K_4\beta_3\epsilon_6\mu_h\phi_1\phi_2\theta_1 + K_4\beta_3\epsilon_6\phi_1^{\,2}\phi_2\theta_1 + K_5\Lambda_h\beta_b\omega_1\phi_1\phi_2\theta_1 \\ +K_5\beta_2\mu_h\omega_1\phi_1\phi_2\theta_1 + K_4K_5\Lambda_h\beta_b\phi_1\phi_2\theta_1 + K_4K_5\beta_1\mu_h\phi_1\phi_2\theta_1 + K_4K_5\beta_1\phi_1^{\,2}\phi_2\theta_1 \\ +K_5\beta_2\omega_1\phi_1^{\,2}\phi_2\theta_1 + \Lambda_h\beta_b\epsilon_1\omega_1\phi_1\phi_2\theta_1 + \beta_3\epsilon_1\mu_h\omega_1\phi_1\phi_2\theta_1 + \beta_3\epsilon_1\omega_1\phi_1^{\,2}\phi_2\theta_1 + \left(1 - R_0^B\right) \end{array} \right) \right). \qquad (25)$$

By applying the Routh-Hurwitz criterion which states that all roots of the polynomial have negative real parts if and only if $M_1 > 0, M_2 > 0, M_3 > 0, M_4 > 0, M_5 > 0, M_6 > 0, M_7 > 0$.

For all these conditions to be satisfied, then $R_0^B < 1$.

Therefore, by Routh-Hurwitz criterion, the disease-free equilibrium of the Hepatitis B virus model (1) is locally asymptotically stable when $R_0^B < 1$.

Global Asymptotic Stability of the Disease-Free Equilibrium Point of the Hepatitis B Model.

To investigate the global stability of the disease-free equilibrium, we use the technique implemented by Castillo-Chavez and song (2004).

To do this, we write the equation in the uninfected class as

$$\frac{dX}{dt} = F(X, Z),$$

And we re-write the equation in the infected class as

$$\frac{dZ}{dt} = G(X, Z),$$

Where $X = (S_h, V_B, R_B) \in R_+^3$ denotes the uninfected population and

$$H_2 : \frac{dZ}{dt} = D_Z G(X^*, 0)Z - \hat{G}(X, Z),$$

denotes the infected population

$\varepsilon_{0H} = (X^*, 0)$ represent the disease-free equilibrium of the system, and it globally asymptotically stable if it satisfies the following conditions:

$H_1 : \dfrac{dX}{dt} = F(X^*, 0)$, X^* is globally asymptotically stable

At Disease free equilibrium

$$\frac{dS_h}{dt} = \Lambda_h - \mu_h S_h,$$

$$H_2 : \frac{dZ}{dt} = D_Z G(X^*, 0)Z - \hat{G}(X, Z), \qquad (26)$$

$\hat{G}(X, Z) \geq 0$ for all $(X, Z) \in D$ and where $D_Z G(X^*, 0)$ is an M- matrix (i.e the diagonal elements are no-negative and it is also the Jacobian of $\hat{G}(X, Z) \geq 0$ evaluated at $(X^*, 0)$.If the system satisfies the above condition, then the theorem below holds.

Theorem 4

The equilibrium point $\varepsilon_{0B} = (X^*, 0)$ *is globally asymptotically stable if* $R_0^B \leq 1$.

From the above, it is obvious that

$$F(X, Z) = \begin{pmatrix} (1 - \beta_b I_A)\Lambda_h + \phi_1 V_B + \gamma R_B \\ -\dfrac{(\beta_1 I_A + \beta_2 I_C + \beta_3 T_B)}{N_h} S_h \\ -(\phi_2 + \mu_h) S_h \\ \phi_2 S_h - (\phi_1 + \mu_h) V_B \\ \tau T_B - (\gamma + \mu_h) R_B \end{pmatrix} = \frac{dX}{dt} \qquad (27)$$

$$G(X, Z) = \begin{pmatrix} \dfrac{(\beta_1 I_A + \beta_2 I_C + \beta_3 T_B)}{N_h} S_h - (\theta_1 + \mu_h) E_B \\ \theta_1 E_B + \beta_b \Lambda_h - (\varepsilon_6 + \omega_1 + \delta_1 + \mu_h) I_A \\ \omega_1 I_A - (\varepsilon_1 + \delta_3 + \mu_h) I_C \\ \varepsilon_6 I_A + \varepsilon_1 I_C - (\tau + \delta_6 + \mu_h) T_B \end{pmatrix} = \frac{dZ}{dt}, \qquad (28)$$

$$D_Z\left(X^*,0\right)=\begin{pmatrix} -(\theta_1+\mu_h) & \dfrac{\beta_1(\phi_1+\mu_h)}{(\phi_2+\phi_1+\mu_h)} & \dfrac{\beta_2(\phi_1+\mu_h)}{(\phi_2+\phi_1+\mu_h)} & \dfrac{\beta_3(\phi_1+\mu_h)}{(\phi_2+\phi_1+\mu_h)} \\ \theta_1 & \beta_b\Lambda_h-(\varepsilon_6+\omega_1+\delta_1+\mu_h) & 0 & 0 \\ 0 & \omega_1 & -(\varepsilon_1+\delta_3+\mu_h) & 0 \\ 0 & \varepsilon_6 & \varepsilon_1 & -(\tau+\delta_6+\mu_h) \end{pmatrix},$$

To prove that

$\hat{G}(X,Z)\geq 0$,

$$D_Z\left(X^*,0\right)Z=\begin{pmatrix} -(\theta_1+\mu_h) & \dfrac{\beta_1(\phi_1+\mu_h)}{(\phi_2+\phi_1+\mu_h)} & \dfrac{\beta_2(\phi_1+\mu_h)}{(\phi_2+\phi_1+\mu_h)} & \dfrac{\beta_3(\phi_1+\mu_h)}{(\phi_2+\phi_1+\mu_h)} \\ \theta_1 & \beta_b\Lambda_h-(\varepsilon_6+\omega_1+\delta_1+\mu_h) & 0 & 0 \\ 0 & \omega_1 & -(\varepsilon_1+\delta_3+\mu_h) & 0 \\ 0 & \varepsilon_6 & \varepsilon_1 & -(\tau+\delta_6+\mu_h) \end{pmatrix}\begin{pmatrix} E_B \\ I_A \\ I_C \\ T_B \end{pmatrix},$$

$$D_Z\left(X^*,0\right)Z=\begin{pmatrix} -(\theta_1+\mu_h)E_B+\dfrac{\beta_1(\phi_1+\mu_h)}{(\phi_2+\phi_1+\mu_h)}I_A+\dfrac{\beta_2(\phi_1+\mu_h)}{(\phi_2+\phi_1+\mu_h)}I_C+\dfrac{\beta_3(\phi_1+\mu_h)}{(\phi_2+\phi_1+\mu_h)}T_B \\ \theta_1E_B+\beta_b\Lambda_h-(\varepsilon_6+\omega_1+\delta_1+\mu_h)I_A \\ \omega_1I_A-(\varepsilon_1+\delta_3+\mu_h)I_C \\ \varepsilon_6I_A+\varepsilon_1I_C-(\tau+\delta_6+\mu_h)T_B \end{pmatrix},$$

$$H_2:\frac{dZ}{dt}=D_ZG\left(X^*,0\right)Z-\hat{G}(X,Z),$$

$$H_2:\hat{G}(X,Z)=D_ZG\left(X^*,0\right)Z-G(X,Z),$$

$$\hat{G}(X,Z)=\begin{pmatrix} \beta_1I_A+\beta_2I_C+\beta_3T_B-(\beta_1I_A+\beta_2I_C+\beta_3T_B)\dfrac{S_h}{N_h} \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\hat{G}(X,Z)=\begin{pmatrix} (\beta_1I_A+\beta_2I_C+\beta_3T_B)\left(1-\dfrac{S_h}{N_h}\right) \\ 0 \\ 0 \\ 0 \end{pmatrix}. \qquad (29)$$

From the above, it is obvious that

$\hat{G}_1(X,Z)\geq 0$ *and* $\hat{G}_2(X,Z)=\hat{G}_3(X,Z)=\hat{G}_4(X,Z)=0,.,$

which implies that $\hat{G}_1(X,Z)\geq 0$.

Consequently, since the given necessary and sufficient conditions stated above is satisfied, then the disease-free equilibrium ε_{0B} of Hepatitis B model is globally asymptotically stable when $R_0^B<1$.

Sensitivity Analysis of Hepatitis B Model

The model parameters are investigated using sensitivity analysis and their relative influence on the basic reproduction number R_0^B is determined. It helps identify which parameters will have the greatest effect on the transmission dynamics of Hepatitis B and thus provides information on which parameters are most important for controlling the disease.

It can be determine using: $S_x^{R_0^B}=\left(\dfrac{\partial R_0^B}{\partial x}\right)\left(\dfrac{x}{R_0^B}\right)$.

$$S_{\theta_1}^{R_0^B}=\frac{\left(\begin{pmatrix}\beta_1\mu_h^{\;2}+\left(\beta_1\epsilon_1+\left(\tau+\delta_3+\delta_6\right)\beta_1+\omega_1\beta_2+\varepsilon_6\beta_3\right)\mu_h\\+\left(\left(\tau+\delta_6\right)\beta_1+\beta_3\left(\omega_1+\varepsilon_6\right)\right)\varepsilon_1\\+\delta_3\left(\tau+\delta_6\right)\beta_1+\delta_3\epsilon_6\beta_3+\omega_1\beta_2\left(\tau+\delta_6\right)\end{pmatrix}\theta_1\left(\mu_h+\phi_1\right)\mu_h\right)}{\begin{aligned}&\Lambda_h\beta_b\mu_h^{\;4}+\left(\left(\Lambda_h\beta_b+\beta_1\right)\theta_1+\Lambda_h\beta_b\left(\tau+\delta_3+\delta_6+\epsilon_1+\phi_1+\phi_2\right)\right)\mu_h^{\;3}\\&+\begin{pmatrix}\Lambda_h\beta_b\left(\tau+\delta_3+\delta_6+\varepsilon_1+\phi_1+\phi_2\right)+\beta_1\phi_1+\beta_1\varepsilon_1+\left(\tau+\delta_3+\delta_6\right)\beta_1\\+\omega_1\beta_2+\varepsilon_6\beta_3\end{pmatrix}\theta_1\\&+\beta_b\left(\left(\tau+\delta_3+\delta_6+\varepsilon_1\right)\phi_1+\left(\tau+\delta_6+\phi_2\right)\varepsilon_1+\left(\tau+\delta_6+\phi_2\right)\delta_3+\phi_2\left(\tau+\delta_6\right)\right)\Lambda_h\mu_h^{\;2}\\&+\begin{pmatrix}\begin{pmatrix}\beta_b\left(\left(\tau+\delta_3+\delta_6+\varepsilon_1\right)\phi_1+\left(\tau+\delta_6+\phi_2\right)\varepsilon_1+\left(\tau+\delta_6+\phi_2\right)\delta_3+\phi_2\left(\tau+\delta_6\right)\right)\Lambda_h\\+\left(\beta_1\varepsilon_1+\left(\tau+\delta_3+\delta_6\right)\beta_1+\omega_1\beta_2+\varepsilon_6\beta_3\right)\phi_1+\left(\left(\tau+\delta_6\right)\beta_1+\beta_3\left(\omega_1+\varepsilon_6\right)\right)\varepsilon_1\\+\delta_3\left(\tau+\delta_6\right)\beta_1+\delta_3\varepsilon_6\beta_3+\omega_1\beta_2\left(\tau+\delta_6\right)\end{pmatrix}\theta_1\\+\Lambda_h\beta_b\left(\phi_2+\phi_1\right)\left(\varepsilon_1+\delta_3\right)\left(\tau+\delta_6\right)\end{pmatrix}\mu_h\\&+\left(\Lambda_h\beta_b\left(\phi_2+\phi_1\right)\left(\varepsilon_1+\delta_3\right)\left(\tau+\delta_6\right)+\begin{pmatrix}\left(\left(\tau+\delta_6\right)\beta_1+\beta_3\left(\omega_1+\varepsilon_6\right)\right)\varepsilon_1+\delta_3\left(\tau+\delta_6\right)\beta_1\\+\delta_3\varepsilon_6\beta_3+\omega_1\beta_2\left(\tau+\delta_6\right)\end{pmatrix}\phi_1\right)\\&\theta_1\left(\theta_1+\mu_h\right)\end{aligned}}=0.24059,$$

$$S_{\varepsilon_1}^{R_0^B}=\frac{-\left(\left(\beta_2-\beta_3\right)\mu_h-\delta_3\beta_3+\beta_2\left(\tau+\delta_6\right)\right)\theta_1\left(\mu_h+\phi_1\right)\omega_1\varepsilon_1}{\begin{aligned}&\Lambda_h\beta_b\mu_h^{\;4}+\left(\left(\Lambda_h\beta_b+\beta_1\right)\theta_1+\Lambda_h\beta_b\left(\tau+\delta_3+\delta_6+\varepsilon_1+\phi_1+\phi_2\right)\right)\mu_h^{\;3}\\&+\begin{pmatrix}\begin{pmatrix}\Lambda_h\beta_b\left(\tau+\delta_3+\delta_6+\varepsilon_1+\phi_1+\phi_2\right)+\tau\beta_1+\beta_1\delta_6+\omega_1\beta_2+\beta_1\varepsilon_1\\+\varepsilon_6\beta_3+\beta_1\phi_1+\beta_1\delta_3\end{pmatrix}\theta_1\\+\beta_b\left(\left(\tau+\delta_3+\delta_6+\varepsilon_1\right)\phi_1+\left(\tau+\delta_6+\phi_2\right)\varepsilon_1+\left(\tau+\delta_6+\phi_2\right)\delta_3+\phi_2\left(\tau+\delta_6\right)\right)\Lambda_h\end{pmatrix}\mu_h^{\;2}\\&+\beta_b\left(\left(\tau+\delta_3+\delta_6+\varepsilon_1\right)\phi_1+\left(\tau+\delta_6+\phi_2\right)\varepsilon_1+\left(\tau+\delta_6+\phi_2\right)\delta_3+\phi_2\left(\tau+\delta_6\right)\right)\Lambda_h\\&+\left(\tau\beta_1+\beta_1\delta_3+\beta_1\delta_6+\beta_1\varepsilon_1+\omega_1\beta_2+\varepsilon_6\beta_3\right)\phi_1++\left(\tau\beta_1+\beta_1\delta_6+\beta_3\left(\omega_1+\varepsilon_6\right)\right)\varepsilon_1\\&+\left(\tau\beta_1+\beta_1\delta_6+\varepsilon_6\beta_3\right)\delta_3+\omega_1\beta_2\left(\tau+\delta_6\right)\theta_1+\Lambda_h\beta_b\left(\phi_2+\phi_1\right)\left(\varepsilon_1+\delta_3\right)\left(\tau+\delta_6\right)\mu_h\\&+\theta_1\left(\Lambda_h\beta_b\left(\phi_2+\phi_1\right)\left(\varepsilon_1+\delta_3\right)\left(\tau+\delta_6\right)+\begin{pmatrix}\left(\tau\beta_1+\beta_1\delta_6+\beta_3\left(\omega_1+\varepsilon_6\right)\right)\varepsilon_1+\left(\tau\beta_1+\beta_1\delta_6+\varepsilon_6\beta_3\right)\delta_3\\+\omega_1\beta_2\left(\tau+\delta_6\right)\end{pmatrix}\phi_1\right)\left(\epsilon_1+\delta_3+\mu_h\right)\end{aligned}}=-0.01155,$$

$$S_{\delta_3}^{R_0^B}=\frac{-\left(\left(\beta_2-\beta_3\right)\mu_h-\delta_3\beta_3+\beta_2\left(\tau+\delta_6\right)\right)\theta_1\left(\mu_h+\phi_1\right)\omega_1\varepsilon_1}{\begin{aligned}&\Lambda_h\beta_b\mu_h{}^4+\left(\left(\Lambda_h\beta_b+\beta_1\right)\theta_1+\Lambda_h\beta_b\left(\tau+\delta_3+\delta_6+\varepsilon_1+\phi_1+\phi_2\right)\right)\mu_h{}^3\\&+\left(\begin{matrix}\left(\begin{matrix}\Lambda_h\beta_b\left(\tau+\delta_3+\delta_6+\varepsilon_1+\phi_1+\phi_2\right)+\tau\beta_1+\beta_1\delta_6+\omega_1\beta_2+\beta_1\varepsilon_1+\varepsilon_6\beta_3\\+\beta_1\phi_1+\beta_1\delta_3\end{matrix}\right)\theta_1\\+\beta_b\left(\left(\tau+\delta_3+\delta_6+\varepsilon_1\right)\phi_1+\left(\tau+\delta_6+\phi_2\right)\varepsilon_1+\left(\tau+\delta_6+\phi_2\right)\delta_3+\phi_2\left(\tau+\delta_6\right)\right)\Lambda_h\end{matrix}\right)\mu_h{}^2\\&+\beta_b\left(\left(\tau+\delta_3+\delta_6+\varepsilon_1\right)\phi_1+\left(\tau+\delta_6+\phi_2\right)\varepsilon_1+\left(\tau+\delta_6+\phi_2\right)\delta_3+\phi_2\left(\tau+\delta_6\right)\right)\Lambda_h\\&+\left(\tau\beta_1+\beta_1\delta_3+\beta_1\delta_6+\beta_1\varepsilon_1+\omega_1\beta_2+\varepsilon_6\beta_3\right)\phi_1+\left(\tau\beta_1+\beta_1\delta_6+\beta_3\left(\omega_1+\varepsilon_6\right)\right)\varepsilon_1\\&+\left(\tau\beta_1+\beta_1\delta_6+\varepsilon_6\beta_3\right)\delta_3+\omega_1\beta_2\left(\tau+\delta_6\right)\theta_1+\Lambda_h\beta_b\left(\phi_2+\phi_1\right)\left(\varepsilon_1+\delta_3\right)\left(\tau+\delta_6\right)\mu_h\\&+\theta_1\left(\Lambda_h\beta_b\left(\phi_2+\phi_1\right)\left(\varepsilon_1+\delta_3\right)\left(\tau+\delta_6\right)+\left(\begin{matrix}\left(\tau\beta_1+\beta_1\delta_6+\beta_3\left(\omega_1+\varepsilon_6\right)\right)\varepsilon_1\\+\left(\tau\beta_1+\beta_1\delta_6+\varepsilon_6\beta_3\right)\delta_3\\+\omega_1\beta_2\left(\tau+\delta_6\right)\end{matrix}\right)\phi_1\right)\left(\varepsilon_1+\delta_3+\mu_h\right)\end{aligned}}=-0.04086,$$

$$S_{\delta_6}^{R_0^B}=\frac{-\left(\varepsilon_6\mu_h+\left(\omega_1+\varepsilon_6\right)\varepsilon_1+\delta_3\varepsilon_6\right)\beta_3\delta_6\theta_1\left(\mu_h+\phi_1\right)}{\begin{aligned}&\left(\tau+\mu_h+\delta_6\right)\Lambda_h\beta_b\mu_h{}^4+\left(\left(\Lambda_h\beta_b+\beta_1\right)\theta_1+\Lambda_h\beta_b\left(\tau+\delta_3+\delta_6+\varepsilon_1+\phi_1+\phi_2\right)\right)\mu_h{}^3\\&+\left(\begin{matrix}\left(\Lambda_h\beta_b\left(\tau+\delta_3+\delta_6+\varepsilon_1+\phi_1+\phi_2\right)+\tau\beta_1+\beta_1\delta_6+\omega_1\beta_2+\beta_1\varepsilon_1+\varepsilon_6\beta_3+\beta_1\phi_1+\beta_1\delta_3\right)\theta_1\\+\beta_b\left(\left(\tau+\delta_3+\delta_6+\varepsilon_1\right)\phi_1+\left(\tau+\delta_6+\phi_2\right)\varepsilon_1+\left(\tau+\delta_6+\phi_2\right)\delta_3+\phi_2\left(\tau+\delta_6\right)\right)\Lambda_h\end{matrix}\right)\mu_h{}^2\end{aligned}}=-0.00953,$$

$$S_{\tau}^{R_0^B}=\frac{-\tau\left(\varepsilon_6\mu_h+\left(\omega_1+\varepsilon_6\right)\varepsilon_1+\delta_3\varepsilon_6\right)\beta_3\theta_1\left(\mu_h+\phi_1\right)}{\begin{aligned}&\left(\tau+\delta_6+\mu_h\right)\Lambda_h\beta_b\mu_h{}^4+\left(\left(\Lambda_h\beta_b+\beta_1\right)\theta_1+\Lambda_h\beta_b\left(\tau+\delta_3+\delta_6+\epsilon_1+\phi_1+\phi_2\right)\right)\mu_h{}^3\\&+\left(\begin{matrix}\left(\Lambda_h\beta_b\left(\tau+\delta_3+\delta_6+\varepsilon_1+\phi_1+\phi_2\right)+\tau\beta_1+\beta_1\delta_6+\omega_1\beta_2+\beta_1\varepsilon_1+\epsilon_6\beta_3+\beta_1\phi_1+\beta_1\delta_3\right)\theta_1\\+\beta_b\left(\left(\tau+\delta_3+\delta_6+\varepsilon_1\right)\phi_1+\left(\tau+\delta_6+\phi_2\right)\varepsilon_1+\left(\tau+\delta_6+\phi_2\right)\delta_3+\phi_2\left(\tau+\delta_6\right)\right)\Lambda_h\end{matrix}\right)\mu_h{}^2\\&+\beta_b\left(\left(\tau+\delta_3+\delta_6+\varepsilon_1\right)\phi_1+\left(\tau+\delta_6+\phi_2\right)\varepsilon_1+\left(\tau+\delta_6+\phi_2\right)\delta_3+\phi_2\left(\tau+\delta_6\right)\right)\Lambda_h\\&+\left(\tau\beta_1+\beta_1\delta_3+\beta_1\delta_6+\beta_1\varepsilon_1+\omega_1\beta_2+\varepsilon_6\beta_3\right)\phi_1++\left(\tau\beta_1+\beta_1\delta_6+\beta_3\left(\omega_1+\varepsilon_6\right)\right)\varepsilon_1\\&+\left(\tau\beta_1+\beta_1\delta_6+\varepsilon_6\beta_3\right)\delta_3+\omega_1\beta_2\left(\tau+\delta_6\right)\theta_1+\Lambda_h\beta_b\left(\phi_2+\phi_1\right)\left(\varepsilon_1+\delta_3\right)\left(\tau+\delta_6\right)\mu_h\\&+\theta_1\left(\Lambda_h\beta_b\left(\phi_2+\phi_1\right)\left(\varepsilon_1+\delta_3\right)\left(\tau+\delta_6\right)+\left(\begin{matrix}\left(\tau\beta_1+\beta_1\delta_6+\beta_3\left(\omega_1+\varepsilon_6\right)\right)\varepsilon_1+\left(\tau\beta_1+\beta_1\delta_6+\varepsilon_6\beta_3\right)\delta_3\\+\omega_1\beta_2\left(\tau+\delta_6\right)\end{matrix}\right)\phi_1\right)\end{aligned}}=-0.0143,$$

$$S_{\Lambda_h}^{R_0^B}=\frac{\begin{pmatrix}(\theta_1+\mu_h)(\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\beta_b\mu_h\\+(\theta_1+\mu_h)(\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\beta_b\phi_1\\+(\theta_1+\mu_h)(\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\beta_b\phi_2\end{pmatrix}\Lambda_h}{\begin{array}{l}(\theta_1+\mu_h)(\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\Lambda_h\beta_b\mu_h+(\theta_1+\mu_h)(\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\Lambda_h\beta_b\phi_1\\+(\theta_1+\mu_h)(\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\Lambda_h\beta_b\phi_2+(\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\beta_1\mu_h\theta_1\\+(\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\beta_1\phi_1\theta_1+(\varepsilon_1+\delta_3+\mu_h)\beta_3\varepsilon_6\mu_h\theta_1+(\varepsilon_1+\delta_3+\mu_h)\beta_3\varepsilon_6\phi_1\theta_1\\+(\tau+\delta_6+\mu_h)\beta_2\mu_h\omega_1\theta_1+(\tau+\delta_6+\mu_h)\beta_2\omega_1\phi_1\theta_1+\beta_3\varepsilon_1\mu_h\omega_1\theta_1+\beta_3\varepsilon_1\omega_1\phi_1\theta_1\end{array}}=0.6257,$$

$$S_{\beta_b}^{R_0^B}=\frac{\begin{pmatrix}(\theta_1+\mu_h)(\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\Lambda_h\mu_h\\+(\theta_1+\mu_h)(\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\Lambda_h\phi_1\\+(\theta_1+\mu_h)(\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\Lambda_h\phi_2\end{pmatrix}\beta_b}{\begin{array}{l}(\theta_1+\mu_h)(\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\Lambda_h\beta_b\mu_h+(\theta_1+\mu_h)(\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\Lambda_h\beta_b\phi_1\\+(\theta_1+\mu_h)(\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\Lambda_h\beta_b\phi_2+(\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\beta_1\mu_h\theta_1\\+(\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\beta_1\phi_1\theta_1+(\varepsilon_1+\delta_3+\mu_h)\beta_3\varepsilon_6\mu_h\theta_1+(\varepsilon_1+\delta_3+\mu_h)\beta_3\varepsilon_6\phi_1\theta_1\\+(\tau+\delta_6+\mu_h)\beta_2\mu_h\omega_1\theta_1+(\tau+\delta_6+\mu_h)\beta_2\omega_1\phi_1\theta_1+\beta_3\varepsilon_1\mu_h\omega_1\theta_1+\beta_3\varepsilon_1\omega_1\phi_1\theta_1\end{array}}=0.5427,$$

$$S_{\phi_1}^{R_0^B}=\frac{\begin{array}{l}(\theta_1+\mu_h)(\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\Lambda_h\beta_b+(\varepsilon_1+\delta_3+\mu_h)\\(\tau+\delta_6+\mu_h)\beta_1\theta_1+(\varepsilon_1+\delta_3+\mu_h)\beta_3\varepsilon_6\theta_1+(\tau+\delta_6+\mu_h)\beta_2\omega_1\theta_1\\+\beta_3\varepsilon_1\omega_1\theta_1\end{array}}{\begin{array}{l}\Lambda_h\beta_b\mu_h{}^4+\left((\Lambda_h\beta_b+\beta_1)\theta_1+\Lambda_h\beta_b(\tau+\delta_3+\delta_6+\varepsilon_1+\phi_1+\phi_2)\right)\mu_h{}^3\\+\left(\begin{array}{l}\begin{pmatrix}\Lambda_h\beta_b(\tau+\delta_3+\delta_6+\varepsilon_1+\phi_1+\phi_2)\\+\tau\beta_1+\beta_1\delta_3+\beta_1\delta_6+\omega_1\beta_2+\beta_1\varepsilon_1+\varepsilon_6\beta_3+\beta_1\phi_1\end{pmatrix}\theta_1\\+\beta_b\begin{pmatrix}(\tau+\delta_3+\delta_6+\varepsilon_1)\phi_1+(\tau+\delta_6+\phi_2)\varepsilon_1+(\tau+\delta_6+\phi_2)\delta_3\\+\phi_2(\tau+\delta_6)\end{pmatrix}\Lambda_h\end{array}\right)\mu_h{}^2\\+\beta_b\left((\tau+\delta_3+\delta_6+\varepsilon_1)\phi_1+(\tau+\delta_6+\phi_2)\varepsilon_1+(\tau+\delta_6+\phi_2)\delta_3+\phi_2(\tau+\delta_6)\right)\Lambda_h\\+(\tau\beta_1+\beta_1\delta_3+\beta_1\delta_6+\beta_1\varepsilon_1+\omega_1\beta_2+\varepsilon_6\beta_3)\phi_1+\left(\tau\beta_1+\beta_1\delta_6+\beta_3(\omega_1+\varepsilon_6)\right)\varepsilon_1\end{array}}=0.03199,$$

$$S_{\phi_2}^{R_0^B} = \frac{-\begin{pmatrix} \mu_h^2\beta_1 + \left(\beta_1\epsilon_1 + (\tau+\delta_3+\delta_6)\beta_1 + \omega_1\beta_2 + \epsilon_6\beta_3\right)\mu_h + \left((\tau+\delta_6)\beta_1 + \beta_3(\omega_1+\epsilon_6)\right)\epsilon_1 \\ +\delta_3(\tau+\delta_6)\beta_1 + \delta_3\epsilon_6\beta_3 + \omega_1\beta_2(\tau+\delta_6) \end{pmatrix}\phi_2\theta_1(\mu_h+\phi_1)}{\begin{aligned} &\Lambda_h\beta_b\mu_h^4 + \left((\Lambda_h\beta_b+\beta_1)\theta_1 + \Lambda_h\beta_b(\tau+\delta_3+\delta_6+\epsilon_1+\phi_1+\phi_2)\right)\mu_h^3 \\ &+\begin{pmatrix} \left(\Lambda_h\beta_b(\tau+\delta_3+\delta_6+\epsilon_1+\phi_1+\phi_2) + \beta_1\phi_1 + \beta_1\epsilon_1 + (\tau+\delta_3+\delta_6)\beta_1 + \omega_1\beta_2 + \epsilon_6\beta_3\right)\theta_1 \\ +\beta_b\left((\tau+\delta_3+\delta_6+\epsilon_1)\phi_1 + (\tau+\delta_6+\phi_2)\epsilon_1 + (\tau+\delta_6+\phi_2)\delta_3 + \phi_2(\tau+\delta_6)\right)\Lambda_h \end{pmatrix}\mu_h^2 \\ &+\begin{pmatrix} \begin{pmatrix} \beta_b\left((\tau+\delta_3+\delta_6+\epsilon_1)\phi_1 + (\tau+\delta_6+\phi_2)\epsilon_1 + (\tau+\delta_6+\phi_2)\delta_3 + \phi_2(\tau+\delta_6)\right)\Lambda_h \\ +\left(\beta_1\epsilon_1 + (\tau+\delta_3+\delta_6)\beta_1 + \omega_1\beta_2 + \epsilon_6\beta_3\right)\phi_1 + \left((\tau+\delta_6)\beta_1 + \beta_3(\omega_1+\epsilon_6)\right)\epsilon_1 \\ +\delta_3(\tau+\delta_6)\beta_1 + \delta_3\epsilon_6\beta_3 + \omega_1\beta_2(\tau+\delta_6) \end{pmatrix}\theta_1 \\ +\Lambda_h\beta_b(\phi_1+\phi_2)(\epsilon_1+\delta_3)(\tau+\delta_6) \end{pmatrix}\mu_h \end{aligned}} = -0.10396,$$

$$S_{\beta_1}^{R_0^B} = \frac{\left((\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\mu_h\theta_1 + (\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\phi_1\theta_1\right)\beta_1}{\begin{aligned} &(\theta_1+\mu_h)(\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\Lambda_h\beta_b\mu_h + (\theta_1+\mu_h)(\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\Lambda_h\beta_b\phi_1 \\ &+(\theta_1+\mu_h)(\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\Lambda_h\beta_b\phi_2 + (\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\beta_1\mu_h\theta_1 \\ &+(\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\beta_1\phi_1\theta_1 + (\varepsilon_1+\delta_3+\mu_h)\beta_3\varepsilon_6\mu_h\theta_1 + (\varepsilon_1+\delta_3+\mu_h)\beta_3\varepsilon_6\phi_1\theta_1 \\ &+(\tau+\delta_6+\mu_h)\beta_2\mu_h\omega_1\theta_1 + (\tau+\delta_6+\mu_h)\beta_2\omega_1\phi_1\theta_1 + \beta_3\varepsilon_1\mu_h\omega_1\theta_1 + \beta_3\varepsilon_1\omega_1\phi_1\theta_1 \end{aligned}} = 0.23058,$$

$$S_{\beta_2}^{R_0^B} = \frac{\left((\tau+\delta_6+\mu_h)\mu_h\omega_1\theta_1 + (\tau+\delta_6+\mu_h)\omega_1\phi_1\theta_1\right)\beta_2}{\begin{aligned} &(\theta_1+\mu_h)(\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\Lambda_h\beta_b\mu_h + (\theta_1+\mu_h)(\epsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\Lambda_h\beta_b\phi_1 \\ &+(\theta_1+\mu_h)(\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\Lambda_h\beta_b\phi_2 + (\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\beta_1\mu_h\theta_1 \\ &+(\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\beta_1\phi_1\theta_1 + (\varepsilon_1+\delta_3+\mu_h)\beta_3\varepsilon_6\mu_h\theta_1 + (\varepsilon_1+\delta_3+\mu_h)\beta_3\varepsilon_6\phi_1\theta_1 \\ &+(\tau+\delta_6+\mu_h)\beta_2\mu_h\omega_1\theta_1 + (\tau+\delta_6+\mu_h)\beta_2\omega_1\phi_1\theta_1 + \beta_3\varepsilon_1\mu_h\omega_1\theta_1 + \beta_3\varepsilon_1\omega_1\phi_1\theta_1 \end{aligned}} = 0.09838,$$

$$S_{\beta_3}^{R_0^B} = \frac{\left((\varepsilon_1+\delta_3+\mu_h)\varepsilon_6\mu_h\theta_1 + (\varepsilon_1+\delta_3+\mu_h)\varepsilon_6\phi_1\theta_1 + \varepsilon_1\mu_h\omega_1\theta_1 + \varepsilon_1\omega_1\phi_1\theta_1\right)\beta_3}{\begin{aligned} &(\theta_1+\mu_h)(\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\Lambda_h\beta_b\mu_h + (\theta_1+\mu_h)(\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\Lambda_h\beta_b\phi_1 \\ &+(\theta_1+\mu_h)(\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\Lambda_h\beta_b\phi_2 + (\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\beta_1\mu_h\theta_1 \\ &+(\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\beta_1\phi_1\theta_1 + (\varepsilon_1+\delta_3+\mu_h)\beta_3\varepsilon_6\mu_h\theta_1 + (\varepsilon_1+\delta_3+\mu_h)\beta_3\varepsilon_6\phi_1\theta_1 \\ &+(\tau+\delta_6+\mu_h)\beta_2\mu_h\omega_1\theta_1 + (\tau+\delta_6+\mu_h)\beta_2\omega_1\phi_1\theta_1 + \beta_3\varepsilon_1\mu_h\omega_1\theta_1 + \beta_3\varepsilon_1\omega_1\phi_1\theta_1 \end{aligned}} = 0.04528,$$

$$S_{\varepsilon_6}^{R_0^B} = \frac{\left((\varepsilon_1+\delta_3+\mu_h)\beta_3\mu_h\theta_1 + (\varepsilon_1+\delta_3+\mu_h)\beta_3\phi_1\theta_1\right)\varepsilon_6}{\begin{aligned} &(\theta_1+\mu_h)(\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\Lambda_h\beta_b\mu_h + (\theta_1+\mu_h)(\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\Lambda_h\beta_b\phi_1 \\ &+(\theta_1+\mu_h)(\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\Lambda_h\beta_b\phi_2 + (\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\beta_1\mu_h\theta_1 \\ &+(\varepsilon_1+\delta_3+\mu_h)(\tau+\delta_6+\mu_h)\beta_1\phi_1\theta_1 + (\varepsilon_1+\delta_3+\mu_h)\beta_3\varepsilon_6\mu_h\theta_1 + (\varepsilon_1+\delta_3+\mu_h)\beta_3\varepsilon_6\phi_1\theta_1 \\ &+(\tau+\delta_6+\mu_h)\beta_2\mu_h\omega_1\theta_1 + (\tau+\delta_6+\mu_h)\beta_2\omega_1\phi_1\theta_1 + \beta_3\varepsilon_1\mu_h\omega_1\theta_1 + \beta_3\varepsilon_1\omega_1\phi_1\theta_1 \end{aligned}} = 0.03641,$$

$$S_{\omega_1}^{R_0^B} = -0.206468, \quad (31) \qquad S_{\mu_h}^{R_0^B} = -0.67721.$$

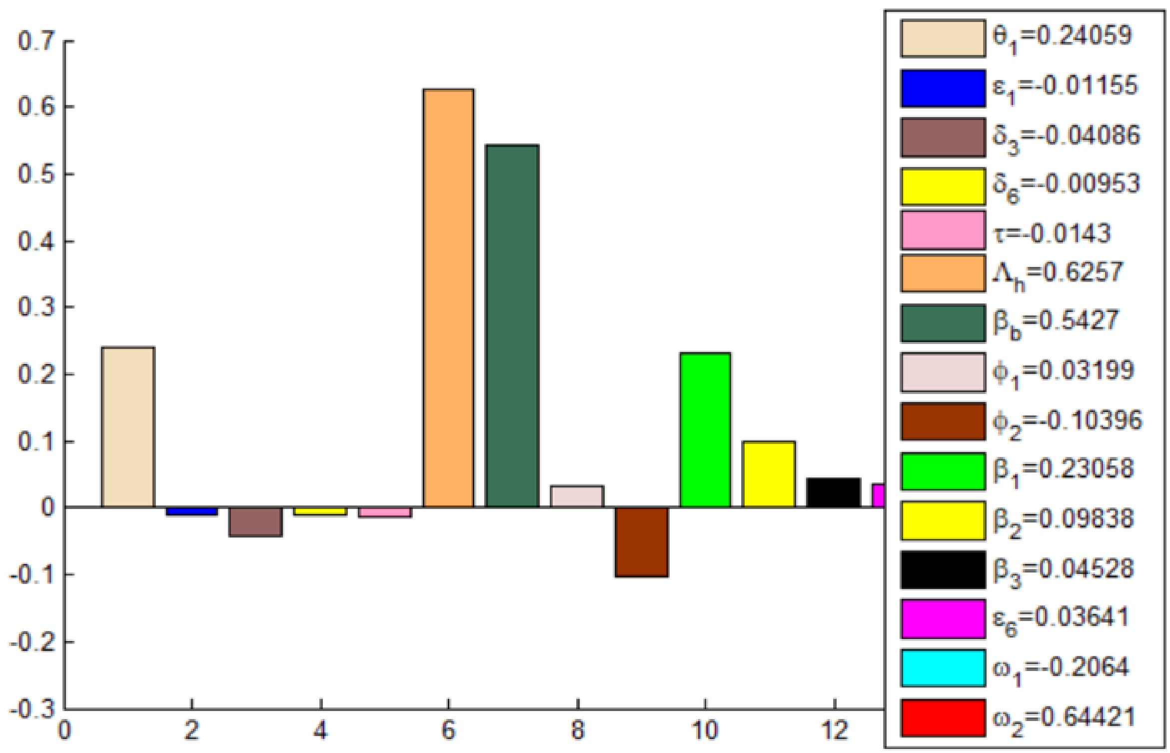

Figure 2: Sensitivity Bar chart of Hepatitis B model

The sensitivity analysis shown in Figure 2 above indicates that the parameters that have positive sensitivity indices are the factors that will increase the transmission and spread of Hepatitis B virus (HBV) in humans. From a biological point of view, positive sensitivity indices imply that when these parameters are increased, the basic reproduction number is increased and thus the risk of HBV transmission is increased. On the other hand, parameters with negative sensitivity indices are decreasing the prevalence and transmission of HBV. From a biological standpoint, the higher these parameters are, the more protective or controlling (vaccination, treatment, recovery, other interventions), thereby decreasing the population of infectives and HBV transmission in humans.

Table 2: Parameter value and sources

Parameter	Value	Source
Λ_h	0.0121	Zhang et al. (2014)
β_1	0.56	Assumed
β_b	0.33	Opoku et al. (2023)
θ_1	0.058426	Ojonimi, et al. (2025)
ε_1	0.025	Zhang et al. (2014)
τ	0.278267	Ojonimi, et al. (2025)
γ	0.045	Zhang et al. (2014)
ϕ_1	0.03	Assumed

ϕ_2	0.1	Assumed
μ_h	0.0121	Zhang et al. (2014)
δ_1	0.01	Assumed
δ_3	0.05	Ojonimi, et al. (2025)
δ_6	0.02	Assumed
ε_6	0.1	Ojonimi, et al. (2025)

RESULTS AND DISCUSSION

Numerical Simulation

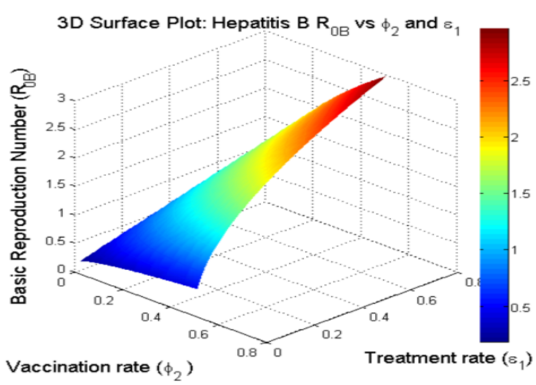

Figure 3: Surface plot showing the Impact of ϕ_2 and ε_1 on R_0^B

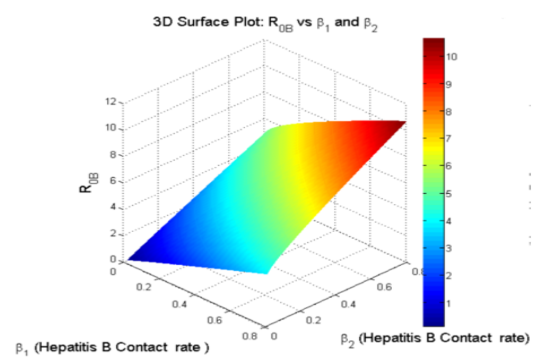

Figure 4: Surface plot showing the Impact of β_2 and β_1 on R_0^B

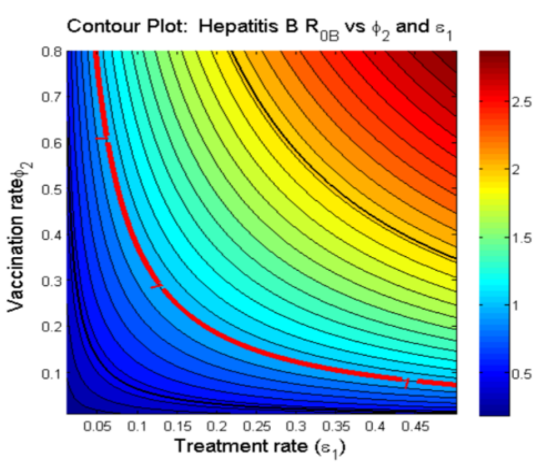

Figure 5: Contour plot showing the Impact of ϕ_2 and ε_1 on R_0^B

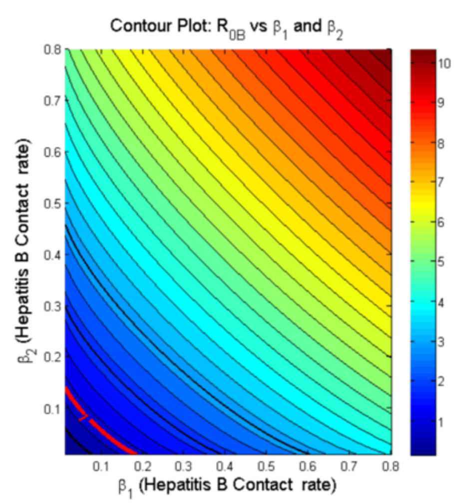

Figure 6: Contour plot showing the Impact of β_2 and β_1 on R_0^B

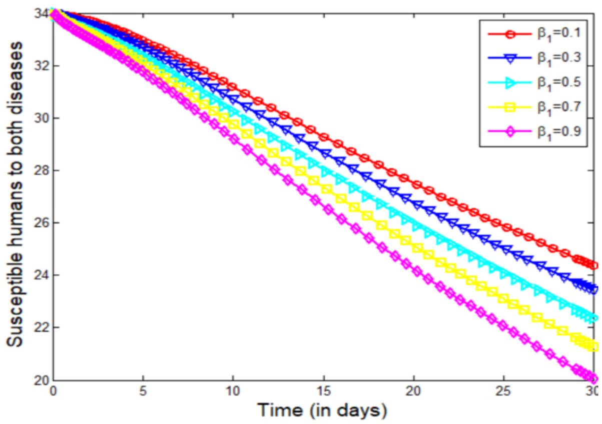

Figure 7: Simulation of the effect of β_1 on susceptible population

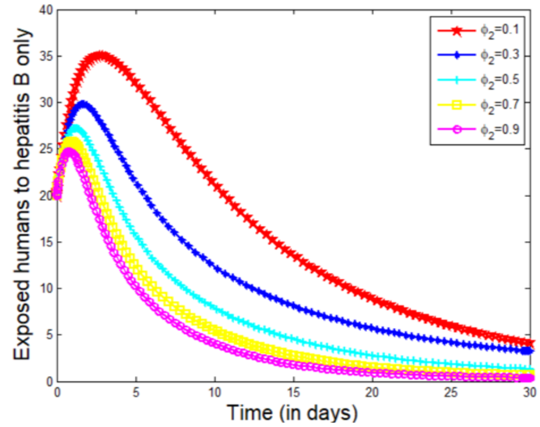

Figure 8: Simulation of the effect of ϕ_2 on exposed human population

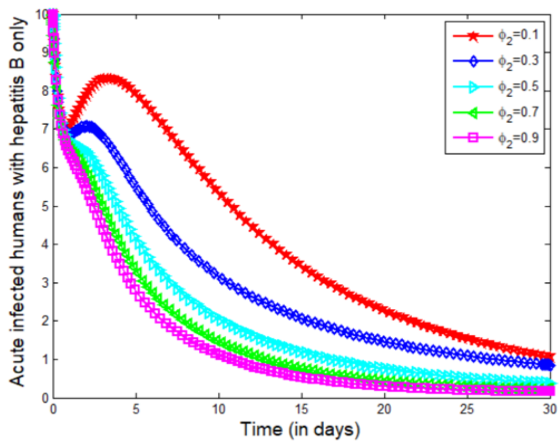

Figure 9: Simulation of the effect of ϕ_2 on acutely infected human population

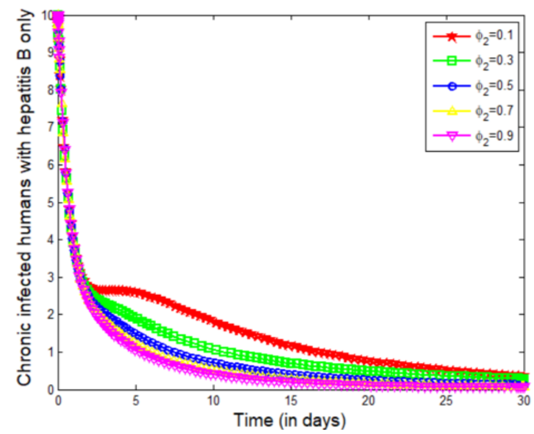

Figure 10: Simulation of the effect of ϕ_2 on chronically infected human population

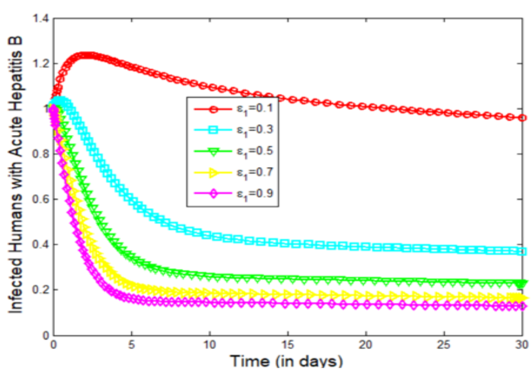

Figure 11: Simulation of the effect of ε_1 on acutely infected human population

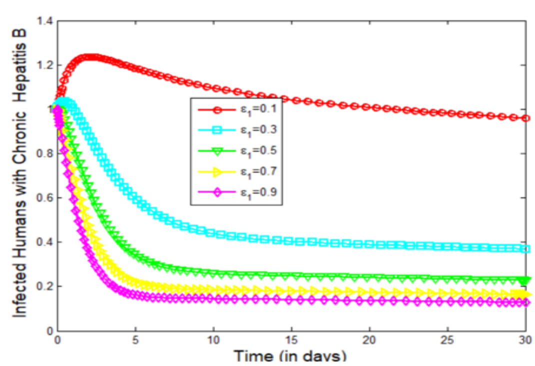

Figure 12: Simulation of the effect of ε_1 on chronically infected human

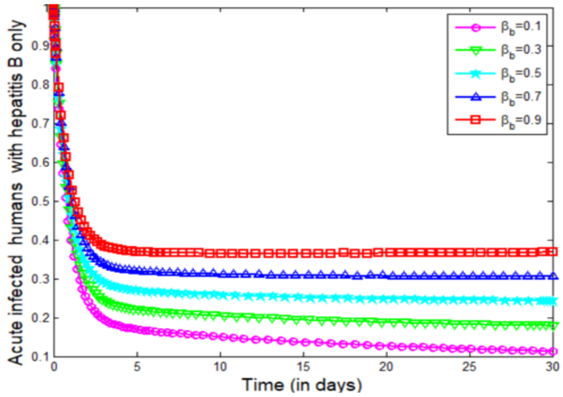

Figure 13: Simulation of the effect of β_b on acutely infected human population

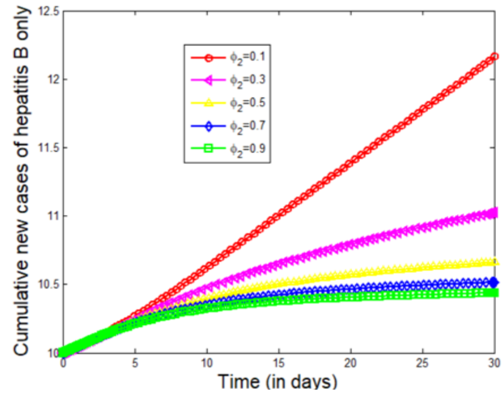

Figure 14: Simulation of the effect of ϕ_2 on cumulative new cases of Hepatitis B

Figure 3 represents the surface plot of the combined effect of the vaccination rate ϕ_2 and treatment rate ε_1 on the basic reproduction number R_0^B of Hepatitis B. Biologically, vaccination would have the effect of reducing the susceptible population and treatment would reduce the infected population, which would ultimately have a dampening effect on disease transmission. Thus, increased efforts in these control measures are expected to lead to a decrease in R_0^B and to improved control of Hepatitis B, as seen in the surface plot in Figure 4, which illustrates the combined effect of Hepatitis B contact rates β_2 and β_1 on the basic reproduction number R_0^B. The surface shown suggests that as β_2 and β_1 increases, so does R_0^B. Increase in contact and transmission rates will further facilitate infection and, in turn, increase transmission potential of Hepatitis B, so reducing and with appropriate preventative measures, reduce and contribute to effective disease control.

In figure 5, the contour plot is displayed, which illustrates combined effects of the vaccination rate ϕ_2 and treatment rate ε_1. on the basic reproduction number R_0^B. The contour lines indicate the values of the basic reproduction number R_0^B of Hepatitis B for combinations of vaccination rate ϕ_2 and treatment rates ε_1. The figure shows that the combined variation of these intervention parameters has a significant impact on the transmission potential of Hepatitis B. The region of parameters where $R_0^B < 1$ is defined by the red threshold contour which approximately corresponds to $R_0^B = 1$. Therefore, combinations of parameters in the region below the threshold represent conditions in which the transmission of Hepatitis B may be controlled, while combinations of parameters in the region above the threshold represent conditions that are favorable for the persistence of the disease.

Fig. 6 shows contour plot of the combined effects of the Hepatitis B contact rates, β_2 and β_1 on the basic reproduction number, R_0^B. These contour lines show the effects of the various combinations of and on the transmission potential of Hepatitis B virus (HBV). As β_2 increases, so does R_0^B and as β_1 increases, so does R_0^B. From a biological standpoint, more contacts lead to more meaningful contacts between susceptible and infected people and boost the spread of HBV in the community. However, as β_2 and β_1 decrease, suggesting that the transmission of HBV can be decreased and eventually controlled. Such interventions to reduce the effective contact and transmission rates, such as health education, safer behaviours, screening and other prevention strategies, can make a significant contribution to the control of the spread of HBV.

Figure (7) depicts the simulation of the effect of the contact rate β_1 on the susceptible human population to Hepatitis B disease. The following is observed: As the contact rate β_1 increases over time, the susceptible human population decreases faster. This means that if the rate of transmission increases, the transmission of Hepatitis B infection is increased, which means that there are fewer susceptible individuals left.

Figure (8) depicts the simulation of the effect of the vaccination rate ϕ_2 on the exposed human population to

Hepatitis B disease. As the proportion ϕ_2 of humans vaccinated increases, the number of exposed human's declines more quickly over time is observed. This means the higher the vaccine coverage, the fewer people will be exposed to Hepatitis B and therefore the more they can help control the spread of the disease.

Figure (9) depicts the simulation of the effect of the vaccination rate ϕ_2 on the acutely infected human population with Hepatitis B. It is revealed that the acutely infected human population decreases with more rapid decay over time, as the vaccination rate ϕ_2 increases. This means higher vaccination coverage means fewer people will be acutely infected and this will help to control the spread of Hepatitis B.

The simulation of the impact of vaccination rate ϕ_2 on the Hepatitis B chronically infected human population is shown in figure (10). We observed that the number of humans infected chronically decreases faster over time as the vaccination rate ϕ_2 increases. This indicates that greater vaccination will be able to make Hepatitis B infection in humans more controllable and less durable.

Figure (11) shows the simulation of the effect of the treatment rate ε_1 on the acutely infected human population with Hepatitis B. As the treatment rate ε_1 increases, it is observed that the human population with acute infection decreases with time faster. This indicates that an increase in the treatment rate ε_1 contributes to reducing the number of acutely infected individuals and helps control the spread of Hepatitis B.

Figure (12) shows the simulated result of the effect of the treatment rate ε_1 on the chronically infected human population of Hepatitis B. It is observed that as the rate of treatment ε_1 increases, the number of chronically infected people decreases more quickly over time. This shows that treating more people is working to reduce the number of people who are chronically infected, and to manage Hepatitis B infection.

The human simulation of the effect of the vertical transmission rate β_b on the acutely infected human population is shown in figure (13). It is noted that the number of acutely infected humans increases with time and tends to a new equilibrium as the vertical transmission rate β_b increases. This means that a higher vertical transmission rate β_b enhances the circulation of Hepatitis B, and raises the number of acutely infected people.

Figure (14) illustrates the modelling of the impact of changing vaccination rate ϕ_2 on cumulative new cases of Hepatitis B. For all values of ϕ_2 it is observed that the cumulative number of new cases grows with time. As vaccination rate ϕ_2 rise, the number of new cases decreases, however. So, the more people vaccinated, the less the new infections of Hepatitis B will occur and it can play a role in controlling the spread of Hepatitis B infection.

CONCLUSION

In this study, we model the dynamics of the transmission of hepatitis B and explore its dynamics including vertical and horizontal transmission with treatment and vaccination as control strategies. We calculated the basic reproduction number R_0^B of the hepatitis B model to show that the system is locally asymptotically stable when $R_0^B < 1$, globally asymptotically stable when $R_0^B < 1$ and a stable endemic equilibrium when $R_0^B > 1$. We performed sensitivity analysis with respect to different parameters such as the vertical transmission rate, contact rates, treatment and vaccination rates. Further, numerical simulations were performed by changing a number of parameters, including the vaccination rate, the treatment rate and the vertical transmission rate. From our simulations we found that despite the natural reduction of vertical transmission rate, the higher the vaccination and treatment rates, the more substantial the alleviation of the hepatitis B burden within the human population. Hence, the importance of strengthening vaccination and treatment programmes, alongside of effective interventions to reduce vertical transmission is emphasised to control and reduce the burden of hepatitis B.

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