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On Common Fixed Point for Finite Family on Maps in a Relational Metric Space.

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ABSTRACT
This research establishes new coincidence and common fixed-point theorems for finite families of self-mappings in complete metric spaces endowed with binary relation. Extending the relational framework initiated by Samet and Turinici and later developed by Shatanawi and Sintunavarat, we introduced g-comparative mappings and prove existence and uniqueness theorems for coincidence and common fixed points of families $\{f_k\}_{k=0}^{m}$ and single map g. The main results are obtained under a generalized contractive inequality $d(f_k x, f_k y) \leq \phi(M_{f_k}(gx, gy))$ where ϕ is comparison function and M_{f_k} is auxiliary metric expression. By assuming relational regularity, weak compatibility, and the existence of an admissible initial point, we construct a modular-iterative sequence and prove its convergence to unique common fixed point. A numerical example is given to describe how the scheme works.

Keywords:
Coincidence point,
Fixed point, and
Relational metric.

INTRODUCTION

For more than a century, fixed point theory in metric spaces has remained a central field of mathematical research, significantly shaping the growth of Functional Analysis as well as many other branches of Mathematics. The concept of a metric space was first introduced by a French mathematician M. Fréchet in 1906. The importance of metric spaces in the advancement of mathematics, especially functional analysis, is immense. Motivated by this foundational idea, many researchers have explored different generalizations of metric spaces in recent years. These include complex-valued metric spaces, rectangular metric spaces, semi-metric spaces, quasimetric spaces, pseudo-metric spaces, probabilistic metric spaces, bi-metric spaces, D-metric spaces, G-metric spaces, S-metric space, cone metric space, partial order metric spaces and relational metric space, among others.

Numerous studies have been devoted to extending and improving fixed point results in various ways; see for example [6] [7] [8] [9] [12] [13] [14] [29] [31] [15] [18] [28] [21] [22] [3]. One remarkable contribution is due to Samet and Turinici who introduced fixed point theorems in metric spaces with binary relation [21]. Later Shatanawi and Sintunavarat [22] established coincidence and common fixed point results in metric spaces. Later, Alam and Imdad introduced another variation of Banach's contraction mapping principle in complete metric spaces equipped with a binary relation, applying relation-theoretic techniques. These works provided analogues of well-known metric notions such as contraction, completeness, and continuity under the universal relation [3].

Parallel to these advances, several works have investigated coincidence and common fixed point results in generalized metric spaces, such as dislocated, symmetric, quasi-metric space and vector-valued metric spaces. The study of set-valued mapping has also expanded the scope of fixed point theory. Petrusel and Petrusel [17] established existence and stability results for coincidence and fixed point problems under weak contractions in vector-valued metric spaces.

Another important direction has been the relaxation of contractive conditions, where results hold even for non-expansive or non-contractive mappings. Pant et al [16] introduced new fixed and common fixed point theorems for a mapping that does not satisfy classical contraction assumptions, with applications to nonlinear integrals. In addition, Thera et al [27] analyzed coincidence and fixed points in symmetric quasi-metric spaces and provided sensitivity analysis for generalized equations, contributing to the stability theory of fixed point problems.

Despite the growing interest in fixed point theory within generalized frameworks, several gaps remain in literature concerning the extension of common fixed point results to families of self-mappings under binary relational constraints. Most existing studies focus on single mappings or pairs of mappings, with limited exploration of systems involving multiple operators.

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This research aims to address this gap by extending coincidence and common fixed-point theorems to finite families of self-mappings.

Definition 0.0.6 Let R be a binary relation on a nonempty set X and $x, y \in X$. We say that x and y are R-comparative if $(x, y) \in R$ or $(y, x) \in R$. We denote it by $[x, y] \in R$ or xSy.

Definition 0.0.7 Let X be a nonempty set and R a binary relation on X. A sequence $\{x_n\} \subset X$ is called R-preserving if $(x_n, x_{n+1}) \in R, \forall n \in \mathbb{N}$.

Definition 0.0.8 [2] Let (X, d) be a metric space and R a binary relation on X. We say that (X, d) is R-complete if every R-preserving Cauchy sequence in X converges to a point in X.

Definition 0.0.9 [2]

- Complete or connected or dichotomous if $[x, y] \in R\ \forall x, y \in X$.
- Weakly complete or weakly connected or trichotomous if $[x, y] \in R$ or $x = y\ \forall x, y \in X$.

Definition 0.0.10 [2] A binary relation R defined on nonempty set X is called

- partial order if R is reflexive, antisymmetric and transitive.
- simple order if R is weakly complete strict order.
- weak order if R is complete preorder.
- total order or linear order or chain if R is complete partial order.
- tolerance if R is reflexive and symmetric.

Definition 0.0.11 [21] Let X be a nonempty set and R a binary relation on X. A subset E of X is called R-directed if for each pair $x, y \in E$, there exists $z \in X$ such that $(x, z) \in R$ and $(y, z) \in R$.

Definition 0.0.12 [2] Let X be a nonempty set and R a binary relation on X. For each $x, y \in X$, a path of length k (where k is a natural number) in R from x to y is a finite sequence $\{z_0, z_1, z_2, \ldots, z_n\} \subset X$ such that:
(i) $z_0 = x$ and $z_k = y$
(ii) $(z_i, z_{i+1}) \in R$ for each i $(0 \leq i \leq k-1)$.

Definition 0.0.13 Let $g: X \to X$ be a mapping. We say a subset M of X is S-g-directed if for every $x, y \in M$, there exists $z \in X$ such that $gxSgz$ and $gySgz$.

Definition 0.0.14 [21] A fundamental result in fixed point theory is the Banach contraction principle asserting that, if (X, d) is a complete metric space then any map $T: X \to X$ satisfying $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$, where $k \in [0,1)$ is a constant, has a unique fixed point.

Definition 0.0.15 Let (X, d) be a metric space endowed with binary relation.
(i) A point $x^* \in X$ is a coincidence point of f_k and g if $f_k(x^*) = g(x^*)\ \forall k \in \{0,1,2, \ldots m\}$.
(ii) A point $x^* \in X$ is a common fixed point of f_k and g if $f_k(x^*) = g(x^*) = x^*\ \forall k \in \{0,1,2, \ldots m\}$.
(iii) Maps f_k and g are said to be weakly compatible if they commute at coincidence point, i.e.

$$f_k(x^*) = g(x^*) \Rightarrow f_k(g(x^*)) = g(f_k(x^*)) \forall k \in \{0,1,2, \ldots m\}.$$

Definition 0.0.16 [21] We say that (X, d, S) is regular if the following condition holds: If the sequence $\{x_n\}$ in X and the point $x^* \in X$ are such that

$$x_n S x_{n+1} \text{ for } n, \text{ and } \lim_{n\to\infty} d(x_n, x^*) = 0,$$

then there exists a subsequence $\{x_{n_p}\}$ of $\{x_n\}$ such that $x_{n_p} Sx$ for all p.

Definition 0.0.17 We say that $f_k: X \longrightarrow X$ is a g-comparative mapping for each $k \in \{0,1,2, \ldots m\}$ if f_k maps g-comparable elements into comparable elements; that is,

$$x, y \in X,\ gxSgy \Longrightarrow f_k x S f_k y.$$

Let ϕ be the set of functions $\phi: [0, \infty) \to [0, \infty)$ satisfying:

$(q_1)\phi$ is nondecreasing;

$(q_2) \sum_{n=1}^{\infty} \phi^n(t) < \infty$ for each $t > 0$, where ϕ^n is the n-th iteration of ϕ.

Lemma 0.0.1 [21] Let $\phi \in \Phi$. We have $\phi(t) < t$ for all $t > 0$.

Proof. Let $\phi \in \Phi$. From property (q2), we have

$$\lim_{n\to\infty} \phi^n(t) = 0, \text{ for all } t > 0. \quad (0.0.1)$$

Suppose that there exists $t > 0$ such that $\phi(t) \geq t$. Since from property (q1), ϕ is nondecreasing, we obtain that

$$\phi^n(t) \geq t, \text{ for all } n \geq 1. \quad (0.0.2)$$

Letting $n \to \infty$ in the above inequality, it follows from above that $t = 0$, which is a contradiction. Then, for all $t > 0$, we have $\phi(t) < t$.

Proposition 0.0.1 If (X, d) is a metric space, R is a binary relation on X, f and g are two self-mappings on X and $\phi \in [0,1)$, then the following contractivity conditions are equivalent:

$$d(fx, fy) \leq \phi d(x, y) \forall x, y \in X \text{ with } (gx, gy) \in R,$$
$$d(fx, fy) \leq \phi d(x, y) \forall x, y \in X \text{ with } [gx, gy] \in R.$$

Proof. The implication (ii) implies (i) is trivial. On the other hand, suppose that (i) holds. Take $x, y \in X$ with $[gx, gy] \in R$. If $(gx, gy) \in R$, then (ii) directly follows from (i). Otherwise, in case $(gy, gx) \in R$, using symmetry of d and (i), we obtain $d(fx, fy) = d(fy, fx) \leq \phi d(gy, gx) = \phi d(gx, gy)$. This proves that (i) implies (ii).

Definition 0.0.18 Let $f_k, g: X \to X$ be two mappings, for each $k \in \{0,1,2, \dots m\}$. For $x, y \in X$,

$$M_{f_k}(gx, gy) = \max\left\{d(gx, gy), \frac{1}{2}[d(gx, f_k x) + d(gy, f_k y)], \frac{1}{2}[d(gx, f_k y) + d(gy, f_k x)]\right\}.$$

Definition 0.0.19 Let $[n] = n mod m$, where

$$[n] = \begin{cases} n & n \leq m \\ k & n > m \end{cases} \& n = mp + k \quad (0.0.3)$$
$$p \in \mathbb{N} \& k \in \{0,1,2, \dots, m\}.$$

Theorem 0.0.1 Let X be a complete metric space and $\{f_k\}_{k=0}^{m}$ be a finite family of self-mappings of X and $g: X \to X$ be a map. Assume that $\phi \in \Phi$ and f_k is g-comparative for each k, then for $x, y \in X, gxSgy$

$$\Rightarrow d(f_k x, f_k y) \leq \phi(M_{f_k}(gx, gy)) \forall k = 0,1,2,3, \dots, m. \quad (0.0.4)$$

Suppose also that the following conditions hold for any $k \in \{0,1,2,3, \dots, m\}$:
(i) $f_k(X) \subseteq g(X)$
(ii) $\lim_{n\to\infty} d(f_{[n+1]}x_{n+1}, f_{[n+2]}x_{n+1}) = 0$
(iii) there exists $x_0 \in X$ such that $gx_0 S f_k x_0$
(iv) (X, d, S) is regular
(v) f_k and g are weakly compatible.
Then f_k and g have a coincidence point $y^* \in X$ for each $k \in \{0,1,2,3, \dots, m\}$. If in addition $gy^* S ggy^*$, then gy^* is a common fixed point of f_k and g for each $k \in \{0,1,2,3, \dots, m\}$.

Proof: Let $x_0 \in X$ such that $gx_0 S f_k x_0$. Since $f_k(X) \subseteq g(X)$ we can choose $x_1 \in X$ so that $gx_1 = f_1 x_0$. Again from $f_k(X) \subseteq g(X)$ we can choose $x_2 \in X$ so that $gx_2 = f_2 x_1$. Continuing this process we can generate a sequence $\{x_n\}$ in X such that $g(x_{n+1}) = f_{[n+1]}(x_n) = y_n, \forall n \geq 0$. It follows from the property of f_k that $\{y_n\}$ is a sequence in X whose consecutive terms are comparable, i.e., $y_n S y_{n+1} \forall n \geq 0$.
Now

$$\begin{aligned} d(y_n, y_{n+1}) &= d(f_{[n+1]}x_n, f_{[n+2]}x_{n+1}) \\ &\leq d(f_{[n+1]}x_n, f_{[n+1]}x_{n+1}) + d(f_{[n+1]}x_{n+1}, f_{[n+2]}x_{n+1}) \\ &\leq \phi(M_{f_{[n+1]}}(gx_n, gx_{n+1})) + d(f_{[n+1]}x_{n+1}, f_{[n+2]}x_{n+1}) \quad (0.0.5) \end{aligned}$$

$$\begin{aligned} M_{f_{[n+1]}}(gx_n, gx_{n+1}) &= \max\left\{d(gx_n, gx_{n+1}), \frac{1}{2}[d(gx_n, f_{[n+1]}x_n) + d(gx_{n+1}, f_{[n+1]}x_{n+1})], \frac{1}{2}[d(gx_n, f_{[n+1]}x_{n+1}) + d(gx_{n+1}, f_{[n+1]}x_n)]\right\} \\ &= \max\left\{d(y_{n-1}, y_n), \frac{1}{2}[d(y_{n-1}, y_n) + d(y_n, y_{n+1})], \frac{1}{2}[d(y_{n-1}, y_{n+1}) + d(y_n, y_n)]\right\} \\ &= \max\left\{d(y_{n-1}, y_n), \frac{1}{2}[d(y_{n-1}, y_n) + d(y_n, y_{n+1})], \frac{1}{2}d(y_{n-1}, y_{n+1})\right\} \\ &\leq \max\left\{d(y_{n-1}, y_n), \frac{1}{2}[d(y_{n-1}, y_n) + d(y_n, y_{n+1})], \frac{1}{2}[d(y_{n-1}, y_n) + d(y_n, y_{n+1})]\right\} \\ &\leq \max\{d(y_{n-1}, y_n), \max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\}\} \\ &= \max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\}. \end{aligned}$$

$$\Rightarrow M_{f_{[n+1]}}(gx_n, gx_{n+1}) \leq \max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\}. \quad (0.0.6)$$

Using equation (0.0.4) and the fact that ϕ is nondecreasing, then $\forall n \geq 1$ we have

$$d(y_n, y_{n+1}) \leq \phi(\max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\}) + d(f_{[n+1]}x_{n+1}, f_{[n+2]}x_{n+1}). \quad (0.0.7)$$

Now, we shall prove that $\{y_n\}$ is a Cauchy sequence in (X, d). If for some $n \geq 0$, we have $y_n = y_{n+1}$, the result follows immediately, so we can suppose that $y_n \neq y_{n+1} \forall n \geq 0$.
Suppose for some $n \geq 1$, we have $d(y_{n-1}, y_n) \leq d(y_n, y_{n+1})$. From (0.0.7)

$$d(y_n, y_{n+1}) \leq \phi[d(y_n, y_{n+1})] < d(y_n, y_{n+1}), \quad (0.0.8)$$

which is a contradiction. Thus $\forall n \geq 1$ we have

$$d(y_{n-1}, y_n) > d(y_n, y_{n+1}). \quad (0.0.9)$$

Then from (0.0.7), for all $n \geq 1$ we have

$$\begin{aligned} d(y_n, y_{n+1}) &\leq \phi(d(y_{n-1}, y_n)) + d(f_{[n+1]}x_{n+1}, f_{[n+2]}x_{n+1}) \\ &\leq \phi^2(d(y_{n-2}, y_{n-1})) + \phi d(f_{[n]}x_n, f_{[n+1]}x_n) + d(f_{[n+1]}x_{n+1}, f_{[n+2]}x_{n+1}) \\ &\leq \phi^3 d(y_{n-3}, y_{n-2}) + \phi^2 d(f_{[n-1]}x_{n-1}, f_{[n]}x_n) \\ &\quad + \phi d(f_{[n]}x_n, f_{[n+1]}x_n) + d(f_{[n+1]}x_{n+1}, f_{[n+2]}x_{n+1}) \\ &\quad \vdots \\ &\leq \phi^n(d(y_{n-n}, y_{n-(n-1)})) + \sum_{i=0}^{n-1} \phi^i\, d(f_{[n+1-i]}x_{n+1-i}, f_{[n+2-i]}x_{n+1-i}) \\ &\leq \phi^n(d(y_0, y_1)) + \sum_{i=0}^{n-1} \phi^i\, d(f_{[n+1-i]}x_{n+1-i}, f_{[n+2-i]}x_{n+1-i}) \quad (0.0.10) \end{aligned}$$

That is,

$$d(y_n, y_{n+1}) \leq \phi^n(d(y_0, y_1)) + \sum_{i=0}^{n-1} \phi^i\, d(f_{[n+1-i]}x_{n+1-i}, f_{[n+2-i]}x_{n+1-i}) \text{for all } n \in \mathbb{N}.$$

Let $n, m \in \mathbb{N}$ with $n > m$. Then

$$\begin{aligned} d(y_m, y_n) &\leq d(y_m, y_{m+1}) + d(y_{m+1}, y_{m+2}) + \cdots + d(y_{n-1}, y_n) \\ &\leq \phi^m(d(y_0, y_1)) + \sum_{i=0}^{m-1} \phi^i\, d(f_{[m+1-i]}x_{m+1-i}, f_{[m+2-i]}x_{m+1-i}) \\ &\quad + \phi^{m+1}(d(y_0, y_1)) + \sum_{i=0}^{m} \phi^i\, d(f_{[m+2-i]}x_{m+1-i}, f_{[m+3-i]}x_{m+2-i}) \\ &\quad + \cdots + \phi^{n-1}(d(y_0, y_1)) + \sum_{i=0}^{n-2} \phi^i\, d(f_{[n-i]}x_{n-i}, f_{[n+1-i]}x_{n-i}) \\ &= \phi^m(d(y_0, y_1))[1 + \phi + \phi^2 + \cdots + \phi^{n-m-1}] \\ &\quad + \sum_{i=0}^{m-1} \phi^i\, d(f_{[m+1-i]}x_{m+1-i}, f_{[m+2-i]}x_{m+1-i}) \\ &\quad + \sum_{i=0}^{m} \phi^i\, d(f_{[m+2-i]}x_{m+1-i}, f_{[m+3-i]}x_{m+2-i}) \\ &\quad + \cdots + \sum_{i=0}^{n-2} \phi^i\, d(f_{[n-i]}x_{n-i}, f_{[n+1-i]}x_{n-i}) \\ &\quad \to 0, m, n \to \infty. \quad (0.0.11) \end{aligned}$$

This implies that $d(y_n, y_m) \to 0$ as $m, n \to \infty$, so that $\{y_n\}$ is Cauchy. Since the metric space (X, d) is complete, then there exists $y^* \in X$ such that

$$\lim_{n\to\infty} d(y_n, y^*) = 0.$$

From (iv), there exists a subsequence $\{y_{n_p}\}$ of $\{y_n\}$ such that y_{n_p} and y^* are comparable for all p. Using (ii), for $p \in \mathbb{N}$ and setting $[n_p + 1] = k$ we have

$$d(gy_{n_p+1}, f_k y^*) = d(f_k y_{n_p}, f_k y^*) \leq \phi\left(M_{f_k}(gy_{n_p}, gy^*)\right). \quad (0.0.12)$$

Now

$$M_{f_k}(gy_{n_p}, gy^*) = \max\left\{d(gy_{n_p}, gy^*), \frac{1}{2}[d(gy_{n_p}, f_k y_{n_p}) + d(gy^*, f_k y^*)], \frac{1}{2}[d(gy_{n_p}, f_k y^*) + d(gy^*, f_k y_{n_p})]\right\}. \quad (0.0.13)$$

Let $p \to \infty$ in (0.0.12). We have

$$\begin{aligned} d(gy^*, f_k y^*) &\leq \phi\left(\max\left\{d(gy^*, gy^*), \frac{1}{2}[d(gy^*, gy^*) + d(gy^*, f_k y^*)], \frac{1}{2}[d(gy^*, f_k y^*) + d(gy^*, gy^*)]\right\}\right) \\ &= \phi\left(\max\left\{0, \frac{1}{2}(0 + d(gy^*, f_k y^*)), \frac{1}{2}[d(gy^*, f_k y^*) + 0]\right\}\right) \\ &< \phi(\max\{0, \max\{0, d(gy^*, f_k y^*)\}\}) = \phi(d(gy^*, f_k y^*)). \end{aligned}$$

Thus

$$d(gy^*, f_k y^*) \leq \phi(d(f_k y^*, gy^*)) < d(gy^*, f_k y^*),$$

which is a contradiction. Therefore,

$$d(gy^*, f_k y^*) = 0,$$

i.e., $gy^* = f_k y^*\ \forall k = 0,1,2, \dots, m$. Hence y^* is a coincidence point of g and f_k.

If in addition, for any $k \in \{0,1,2,3, \dots, m\}$

$$gy^* S ggy^* \Rightarrow f_k y^* S gy^*.$$

Thus

$$d(f_k y^*, f_k gy^*) \leq \phi(M^*_{f_k}(y^*, gy^*)),$$

where

$$\begin{aligned} M^*_{f_k}(y^*, gy^*) &= \max\left\{d(gy^*, ggy^*), \frac{1}{2}[d(gy^*, f_k y^*) + d(gy^*, f_k gy^*)], \frac{1}{2}[d(gy^*, f_k gy^*) + d(ggy^*, f_k y^*)]\right\} \\ &= \max\left\{d(f_k y^*, gf_k y^*), \frac{1}{2}[0 + d(f_k y^*, f_k gy^*)], \frac{1}{2}[d(f_k y^*, f_k gy^*) + d(gf_k y^*, f_k y^*)]\right\} \\ &= \max\left\{d(f_k y^*, f_k gy^*), \frac{1}{2}d(f_k y^*, f_k gy^*), \frac{1}{2}[d(f_k y^*, f_k gy^*) + d(f_k gy^*, f_k y^*)]\right\} \\ &= d(f_k y^*, f_k gy^*). \quad (0.0.14) \end{aligned}$$

Then

$$d(f_k y^*, f_k gy^*) \leq \phi(d(f_k y^*, f_k gy^*)) < d(f_k y^*, f_k gy^*).$$

A contradiction, hence

$$d(f_k y^*, f_k gy^*) = 0 \forall k = 0,1,2, \dots, m.$$
$$\Rightarrow f_k y^* = f_k gy^* \forall k = 0,1,2, \dots, m.$$

i.e., $f_k y^* = f_k f_k y^*$.

Let $z = f_k y^*$. Then $z = f_k z$, i.e., z is a fixed point of $f_k\ \forall k \in \{0,1,2,3, \dots, m\}$. For uniqueness: suppose $M := \bigcap_{k=0}^{m} \mathrm{Fix}\,(f_k) \neq \emptyset$. Let $x^*, y^* \in M$, i.e., $x^* = f_k x^*$ and $y^* = f_k y^*\ \forall k = 0,1,2, \dots, m$. Using (0.0.4):

$$d(f_k x^*, f_k y^*) = d(x^*, y^*) \leq \phi(M_{f_k}(x^*, y^*)),$$

$$\begin{aligned} \text{where } M_{f_k}(x^*, y^*) &= \max\left\{d(x^*, y^*), \frac{1}{2}[d(x^*, f_k x^*) + d(y^*, f_k y^*)], \frac{1}{2}[d(x^*, f_k y^*) + d(y^*, f_k x^*)]\right\} \\ &= \max\left\{d(x^*, y^*), 0, \frac{1}{2}[d(x^*, y^*) + d(y^*, x^*)]\right\} \\ &\leq \max\{d(x^*, y^*), d(x^*, y^*)\} = d(x^*, y^*). \quad (0.0.8) \end{aligned}$$

Now,

$$d(x^*, y^*) \leq \phi(d(x^*, y^*)) < d(x^*, y^*). \quad (0.0.7)$$

This contradiction implies $x^* = y^*$. Hence the common fixed point is unique.

MATERIALS AND METHODS

Examples Illustrating the Theorem

Example 0.1.1 Setting

Metric space: $X = [0,1]$ with the usual metric $d(x, y) = | x - y |$; complete.

Relation: S is the universal relation: xSy for all $x, y \in X$. Then (X, d, S) is regular because every pair is comparable, and any convergent sequence has a subsequence (in fact the whole sequence) comparable with the limit.

Maps:

- $g: X \to X$ defined by $g(x) = x$ (identity).
- For a given integer $m \geq 1$, define $m + 1$ maps:

$$f_k(x) = \frac{x}{k+2}, k = 0,1, \dots, m.$$

So $f_0(x) = \frac{x}{2}, f_1(x) = \frac{x}{3}, \dots, f_m(x) = \frac{x}{m+2}$.

Comparison function: $\phi(t) = \frac{t}{2}$.
Since S is universal, $gxSgy$ always holds. For any k:

$$d(f_k x, f_k y) = \frac{|x - y|}{k + 2} \le \frac{|x - y|}{2}.$$

Because $M_{f_k}(gx, gy) = M_{f_k}(x, y) \ge d(x, y)$, we have

$$d(f_k x, f_k y) \le \frac{1}{2} M_{f_k}(x, y) = \phi(M_{f_k}(x, y)).$$

Hence each f_k is g-comparative.

$$f_k(X) = \left[0, \frac{1}{k + 2}\right] \subseteq [0,1] = g(X).$$

Choose $x_0 = 1$. Define the sequence $\{x_n\}$ cyclically by

$$x_{n+1} = f_{[n+1]}(x_n), \text{where } [n] = n mod(m + 1).$$

Then $g(x_{n+1}) = f_{[n+1]}(x_n)$. Observe that each map is a contraction with factor at most $\frac{1}{2}$, so $x_n \to 0$. Now

$$d(f_{[n+1]}x_{n+1}, f_{[n+2]}x_{n+1}) = |f_{[n+1]}(x_{n+1}) - f_{[n+2]}(x_{n+1})| = |c_{[n+1]} - c_{[n+2]}| x_{n+1},$$

where $c_k = \frac{1}{k+2}$. Since $x_{n+1} \to 0$, the limit is zero.
With $x_0 = 1$, we have $gx_0 = 1$ and $f_k x_0 = \frac{1}{k+2}$.
Because S is universal, $gx_0 S f_k x_0$ for every k.
Regularity of (X, d, S): Immediate from universality of S.
Weak compatibility: The only coincidence point of f_k and g is 0 (since $f_k(0) = 0 = g(0)$). At 0,

$$f_k(g(0)) = f_k(0) = 0 = g(0) = g(f_k(0)),$$

so f_k and g are weakly compatible.

RESULTS AND DISCUSSION

Example 0.1.2 Example 1: Universal Relation

Table 1: Sequence Table (Universal Relation)

n	x_n	g x_n	f_{n+1} x_n	distance
1	1.000000	1.000000	0.500000	0.083333
2	0.500000	0.500000	0.166667	0.013889
3	0.166667	0.166667	0.041667	0.002500
4	0.041667	0.041667	0.008333	0.002083
5	0.008333	0.008333	0.004167	0.000694
6	0.004167	0.004167	0.001389	0.000116
7	0.001389	0.001389	0.000347	0.000017
8	0.000347	0.000347	0.000069	0.000021
9	0.000069	0.000069	0.000035	0.000006
10	0.000035	0.000035	0.000012	0.000001

1.1 Example 2: Non-trivial Partial Order

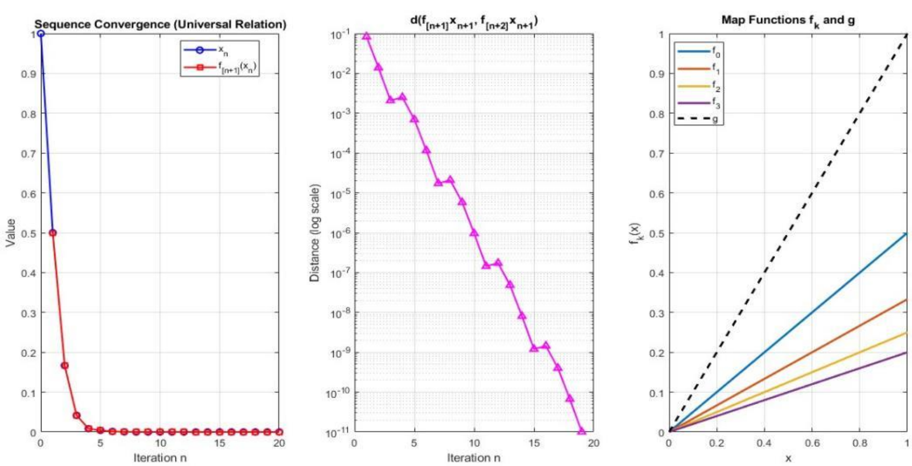

Figure 1: My picture tafida1

Example 0.1.3 Setting

- Metric space: $X = [0,1]$ with the usual metric.
- Relation: Define S by xSy iff $x \ge y$. (X, d, S) is regular: if a sequence $\{a_n\}$ converges to a and is decreasing (i.e., $a_n \ge a_{n+1}$), then $a_n \ge a$ for

all n; such a subsequence can always be selected because any sequence in $[0,1]$ has a monotone subsequence.

Maps:

- $g(x) = x^2$.
- For a fixed $m \geq 1$, define $f_k(x) = \frac{x^2}{k+2}$ for $k = 0,1,\dots,m$. So $f_0(x) = \frac{x^2}{2}, f_1(x) = \frac{x^2}{3}, \dots$.
- Comparison function: $\phi(t) = \frac{t}{2}$.

1. Contraction condition: Assume $gxSgy$, i.e., $x^2 \geq y^2 \Rightarrow x \geq y$ (since $x, y \geq 0$). For any k:

$$d(f_k x, f_k y) = \frac{x^2 - y^2}{k+2} \leq \frac{x^2 - y^2}{2}.$$

Now $M_{f_k}(gx, gy) = M_{f_k}(x^2, y^2)$. Because $x^2 \geq y^2$, the first term in the maximum is $x^2 - y^2$. Hence $M_{f_k}(x^2, y^2) \geq x^2 - y^2$, and

$$d(f_k x, f_k y) \leq \frac{1}{2} M_{f_k}(x^2, y^2) = \phi(M_{f_k}(x^2, y^2)).$$

Thus (0.0.1) holds.

2. Range inclusion: $f_k(X) = \left[0, \frac{1}{k+2}\right] \subseteq [0,1] = g(X)$.
3. Sequence condition: Choose $x_0 = 1$. Define $\{x_n\}$ recursively by

$$g(x_{n+1}) = f_{[n+1]}(x_n), [n] = n \, mod (m+1),$$

i.e., $x_{n+1} = \sqrt{f_{[n+1]}(x_n)}$. For instance,

$$x_0 = 1, g(x_0) = 1,$$
$$x_1 = \sqrt{f_0(1)} = \sqrt{1/2},$$
$$x_2 = \sqrt{f_1(x_1)} = \sqrt{\frac{1/2}{3}} = \sqrt{1/6},$$
$$x_3 = \sqrt{f_0(x_2)} = \sqrt{\frac{1/6}{2}} = \sqrt{1/12}, \dots$$

Clearly $x_n \to 0$. Now compute

$$d(f_{[n+1]}x_{n+1}, f_{[n+2]}x_{n+1}) = \left| \frac{x_{n+1}^2}{c_{[n+1]}} - \frac{x_{n+1}^2}{c_{[n+2]}} \right| = \left| c_{[n+1]}^{-1} - c_{[n+2]}^{-1} \right| x_{n+1}^2,$$

where $c_k = k + 2$. Since $x_{n+1}^2 = f_{[n+1]}(x_n) \to 0$, the limit is zero.

4. Initial point condition: With $x_0 = 1$, $gx_0 = 1$ and $f_k x_0 = \frac{1}{k+2}$. Because $1 \geq \frac{1}{k+2}$, we have $gx_0 S f_k x_0$ for all k.
5. Regularity: The sequence $\{g(x_n)\} = \{x_n^2\}$ is decreasing (since each f_k reduces the square). Hence for the whole sequence we have $g(x_n) \geq g(x_{n+1})$, i.e., $g(x_n) S g(x_{n+1})$. Moreover, $g(x_n) \to 0$, and because the sequence is decreasing, $g(x_n) \geq 0$ for all n, so $g(x_n) S 0$. Thus regularity holds.
6. Weak compatibility: The coincidence points satisfy $f_k(y) = g(y)$, i.e., $\frac{y^2}{k+2} = y^2$, which gives either $y = 0$ or $\frac{1}{k+2} = 1$ (impossible). Hence the only coincidence point is $y^* = 0$. At 0,

$$f_k(g(0)) = f_k(0) = 0 = g(0) = g(f_k(0)),$$

so f_k and g are weakly compatible.

Remark 0.1.4 Both examples illustrate the theorem's conclusion with a finite family of maps, different relations S, and a non-identity map g. The proofs are self-contained and explicitly verify every hypothesis.

Example 0.1.5 Example 2: Partial Order ($x \geq y$)

Table 2: Sequence Table (Partial Order)

n	x_n	g x_n	f_{n+1} x_n	distance
1	1.000000	1.000000	0.500000	0.083333
2	0.707107	0.500000	0.166667	0.013889
3	0.408248	0.166667	0.041667	0.006834
4	0.204124	0.041667	0.008333	0.002500
5	0.091287	0.008333	0.004167	0.000694
6	0.064550	0.004167	0.001389	0.000116
7	0.037680	0.001389	0.000347	0.000017
8	0.018634	0.000347	0.000069	0.000021
9	0.008330	0.000069	0.000035	0.000006
10	0.005893	0.000035	0.000012	0.000001

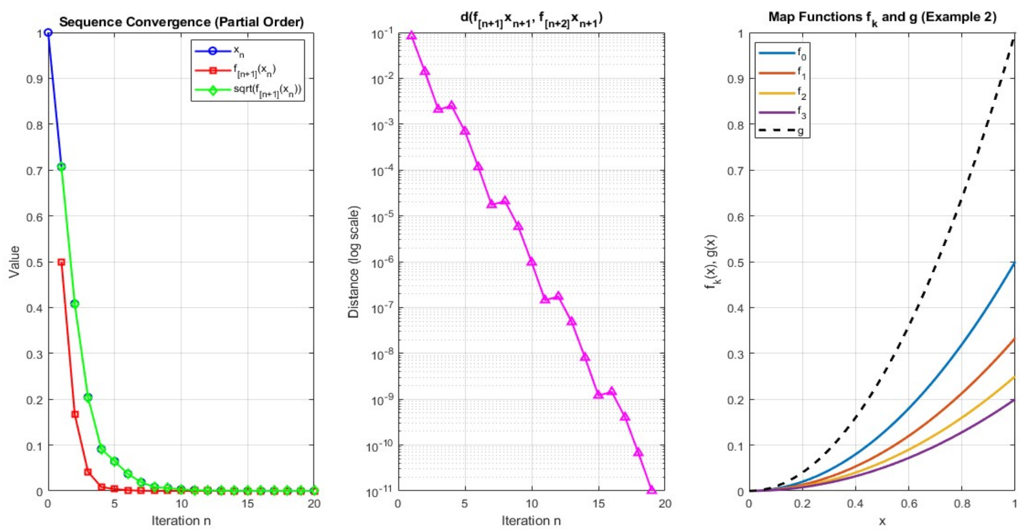

Figure 2: My picture tafida2

Table 3: Verification of contraction condition (Universal Relation example)

k	f_k x	f_k y	d(f_k x, f_k y)	M_{f_k}	φ(M)	Condition hold
0	0.3500	0.1500	0.200000	0.400000	0.200000	true
1	0.2333	0.1000	0.133333	0.400000	0.200000	true
2	0.1750	0.0750	0.100000	0.400000	0.200000	true
3	0.1400	0.0600	0.080000	0.400000	0.200000	true

Table 4: Partial order example ($gx \geq gy$ since $0.7^2 \geq 0.3^2$)

k	f_k x	f_k y	d(f_k x, f_k y)	M_{f_k}	φ(M)	Condition hold
0	0.2450	0.0450	0.200000	0.400000	0.200000	true
1	0.1633	0.0300	0.133333	0.400000	0.200000	true
2	0.1225	0.0225	0.100000	0.400000	0.200000	true
3	0.0980	0.0180	0.080000	0.400000	0.200000	true

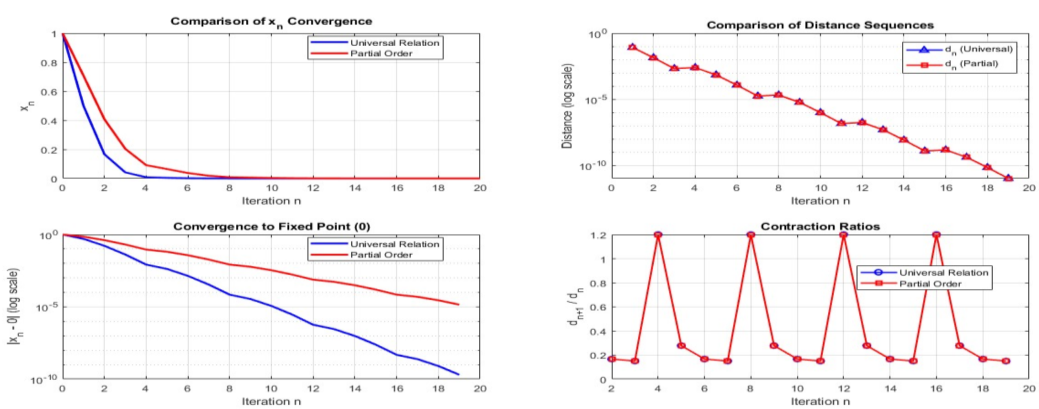

Figure 3: My picture tafida3

CONCLUSION

All assumptions are satisfied. Therefore, for each k, f_k and g have a coincidence point; indeed $y^* = 0$ satisfies $f_k(0) = 0 = g(0)$. Moreover, $g(0) = 0$ and $0Sg(g(0)) = 0$, so 0 is a common fixed point of all f_k and g.

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