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A Multiset-Theoretic Framework for Inventory Optimization in Motor Garage Systems

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ABSTRACT
Inventory system in motor garages involve repeated spare parts such as spark plugs, filters, and bolts, where multiplicity is essential for accurate representation. Classical set-theoretic model fails to capture repeated elements structurally, requiring external quantity variables. This paper develops a multiset-theoretic framework for motor garage inventory modelling, representing inventory as multiplicity functions over a finite universe of items. Algebraic operations, such as addition, difference, union, and intersection are formalized. Structural properties such as closure, commutativity, and stability are established. A dynamic inventory update model and demand-balancing optimization are proposed. Computational analysis demonstrates linear scalability. The framework provides a robust, computationally efficient method for small- and medium- scale inventory management.

INTRODUCTION

Motor garage inventories consist of spare parts such as sparks plugs, oil filters, bolts, bearings, brake pads, and engine components that frequently occur in multiple copies. Efficient management of such inventories is essential for reducing operational costs, preventing stock shortages, and ensuring timely service delivery. Traditional inventory management techniques including the Economic Order Quantity (EOQ) model introduced by Harris (1913), stochastic inventory model developed by Arrow, Karlin, and Scarf (1958), integer-programming-base inventory formulations discussed by Silver, Pyke, and Peterson (1998), and supply-chain optimization frameworks proposed by Simchi-Levi, et.al (2008), have been widely applied in inventory control and logistics. However, these approaches generally treat item quantity as an external numerical parameter rather than an intrinsic structural property of the inventory system.

Classical set theory, introduced by cantor (1874), represents collections as set whose elements are distinct and non-repetive. Consequently, classical sets are inadequate for modelling inventory environments in which identical items occur repeatedly. To address this limitation, multiset theory extends the classical notion of a set by permitting multiple occurrences of the same element. The multiplicity of each element is represented through a counting function, thereby providing a natural mathematical framework for describing collections with repeated objects.

Since it is formal development, multiset theory has evolved into an active area of research with applications in computer science, algebra, information systems optimization, and decision-making. Foundational contributions by Knuth (1981), Blizard (1989), Syropoulos (2001) and Wildberger (2003) established the theoretical basis of multisets and their operations. More recently, William, et.al (2021) introduced the notion of dynamic multisets and integrated their operational properties, demonstrating the suitability of multisets for modelling systems whose contents change over time. Likewise, Ejegwa and Ibrahim (2017) studied comultisets and factor multigroups, extending multisets theory into richer algebraic structures and highlighting its versatility in representing complex real-world systems. Morerecently, Peter et al. (2025) studied substructures and applicability of multiset-based algebraic structures. These developments provide a strong theoretical foundation for extending multiset theory to practical optimization problems such as motor garage inventory management.

These developments suggest that multiset-based model provide a more natural representation of inventory systems than conventional set-theoretic approaches. In a motor garage environment, inventory updates resulting from purchases, sale, usage, replacement, or restocking inherently involve changes in multiplicities. Such changes can be modelled directly through multiset operations without introducing additional quantity variables.

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Consequently, multiset algebra offers both conceptual simplicity and computational efficiency.

Motivated by these observations, this paper develops a multiset-theoretic framework for inventory optimization in motor garage systems. Inventory items are represented as multisets over a finite universe of spare parts, and fundamental inventory processes are formalized through multisets operations. Develops a dynamic inventory update model, and formulates a demand-balancing optimization mechanism suitable for practical operations. The proposed framework integrates multiset algebra with established mathematical modelling techniques to provide a formal representation of inventory systems in motor garage operations.

MATERIALS AND METHODS

This section presents the basic concepts and notation of multiset theory used throughout this paper. For brevity, the term mset is used as an abbreviation for multiset. Inventory items in a motor garage are represented as elements of a finite universe X, while stock quantities are represented through multiplicity functions. Fundamental inventory activities such as restocking, sales, replacement, and stock comparison are modelled using multiset operations including addition, difference, union, and intersection. A dynamic inventory update model is then formulated to describe inventory evolution over time, and an optimization model is introduced to minimize the discrepancy between available stock and forecast demand.

Let $X = \{x_1, x_2, \ldots, x_n\}$ denote the finite universe of inventory items in the motor garage.

Definition (mset) Debnath and Debnath, (2019).

A collection of elements which are allowed to repeat is called a mset. Formally if $\mathcal{S}$ is a set of elements, an mset A drawn from the set $\mathcal{S}$ is represented by a function $C_A\colon \mathcal{S} \to \mathbb{N}$, where $\mathbb{N}$ represents the set of non-negative integers. For each $x \in \mathcal{S}$, $C_A(x)$ is the characteristic value of x in A and indicates the number of occurrences of the element x in A. A mset A is a set if

$$C_A(x) \in \{0,1\}\ \forall x \in \mathcal{S}.$$

The symbolic representation $x \in^k A$ means x belongs to A exactly k times.

Here, we presents mset M drawn from the set $X = \{x_1, x_2, x_3, \ldots x_n\}$ as

$M = \{x_1, x_2, x_3, \ldots x_n\}_{m_1, m_2, m_3, \ldots, m_n}$, or $M = \{m_1/x_1, m_2/x_2, \ldots, m_n/x_n\}$where m_i are the multiplicities of the elements x_i, $i = 1,2, \ldots, n$

Example

Let $X = \{a, b, c, d, e\}$. Then $M = \{a, b, d, e\}_{2,4,5,1} = \{2/a, 4/b, 5/d, 1/e\}$ is a mset drawn from X.

Definition (Cardinality of a mset) Debnath and Debnath, (2019).

The cardinality of a mset A drawn from a set $\mathcal{S}$ denoted by card A is defined by

card $A = \sum_{x \in \mathcal{S}} C_A(x)$. It is also denoted by $|A|$. A mset A over the set $\mathcal{S}$ is said to be finite if $|A| < \infty$. We denote the class of all finite msets over $\mathcal{S}$ by $\mathfrak{M}(\mathcal{S})$.

Definition (root set) Debnath and Debnath, (2019).

Let $A \in \mathfrak{M}(\mathcal{S})$. Then the root set or support set of A denoted A^* is given by the expression

$A^* = \{x \in \mathcal{S} | C_A(x) > 0\}$.For instance, if $A = [a, a, b, b, b, c, c, c, c]$, then $A^* = \{a, b, c\}$. For every x such that $C_A(x) = 0$ implies $x \notin A$.

Note that $x \in A \leftrightarrow C_A(x) > 0$ and $x \in A \leftrightarrow x \in A^*$

Definition (empty mset) Petrovsky, (2004).

Let $A \in \mathfrak{M}(\mathcal{S})$. Then A is said to be empty mset if $C_A(x) = 0\ \forall x \in \mathcal{S}$. We denote the empty mset by $\emptyset$ and $C_\emptyset(x) = 0\ \forall x \in \mathcal{S}$

Definition (equality of msets) Tella et. al., (2024).

Two msets $A, B \in \mathfrak{M}(\mathcal{S})$ are equal denoted by $A \doteqdot B$ if $C_A(x) = C_B(x)\ \forall x \in \mathcal{S}$.

Definition (submset) Tella et. al., (2024).

Let $A, B \in \mathfrak{M}(\mathcal{S})$. The mset A is a submultiset (submset for short) of B denoted by $A \sqsubseteq B$ or $B \sqsupseteq A$ if $C_A(x) \le C_B(x)\ \forall x \in \mathcal{S}$.

If $A \sqsubseteq B$ and $A \neq B$, then A is called the proper submset of B denoted $A \sqsubset B$

Definition (mset operations) Tella et. al., (2024).

i. mset addition

Let $A, B \in \mathfrak{M}(\mathcal{S})$. The mset addition denoted by $A \oplus B$ is given by

$$C_{A \oplus B}(x) = C_A(x) + C_B(x)\ \forall x \in \mathcal{S}$$

ii. set difference

Let $A, B \in \mathfrak{M}(\mathcal{S})$.The difference of B from A denoted by $A \ominus B$ is given by

$$C_{A \ominus B}(x) = Max\{C_A(x) - C_B(x), 0\}\ \forall x \in \mathcal{S}.$$

iii. mset Union

Let $A, B \in \mathfrak{M}(\mathcal{S})$. Then the union of A and B denoted by $A \uplus B$ is defined by

$$C_{A \uplus B}(x) = max\{C_A(x), C_B(x)\}\ \forall x \in \mathcal{S}.$$

iv. mset Intersection

Let $A, B \in \mathfrak{M}(\mathcal{S})$.Then the intersection of A and B denoted $A \cap B$ is defined by

$$C_{A \cap B}(x) = min\{C_A(x), C_B(x)\} \forall x \in \mathcal{S}$$

v. Symmetric Difference

Let $A, B \in \mathfrak{M}(\mathcal{S})$.Then the symmetric difference of A and B denoted $A \bar{\Delta} B$ is defined by $C_{A \bar{\Delta} B}(x) = |C_A(x) - C_B(x)|$.

Note that $A \bar{\Delta} B = (A \ominus B) \uplus (B \ominus A)$

vi. mset complement

Let $G = \{A_1, A_2, A_3, \ldots, A_n | A_i \in \mathfrak{M}(\mathcal{S})\}$ and $Z = \uplus A_i$. Then the complement for each $A \in G$ denoted by $\bar{A}$ is defined by

$C_{\bar{A}}(x) = C_Z(x) - C_A(x) \forall x \in \mathcal{S}$

Note that for each $A \in G$, we have $A \sqsubseteq Z$ and $A \cap \bar{A} \neq \emptyset$ in general

vii. mset scalar multiplication

$A \in \mathfrak{M}(\mathcal{S})$ and $\propto \in \mathbb{Z}^+$. The scalar multiplication of an mset A denoted $\propto \odot A$

is defined by $C_{\propto \odot A} = \propto \cdot C_A(x), \forall x \in \mathcal{S},\ \propto \in \mathbb{Z}^+$

viii. mset arithmetic multiplication

Let $A, B \in \mathfrak{M}(\mathcal{S})$. Then the arithmetic multiplication of A and B denoted by $A \odot B$ is defined by $C_{A \odot B}(x) = C_A(x) \cdot C_B(x)\ \forall x \in \mathcal{S}$

ix. raising to arithmetic power

Let $A \in \mathfrak{M}(\mathcal{S})$. Then the mset A raising to arithmetic power of n denoted by A^n is defined by $C_{A^n}(x) = \left(C_A(x)\right)^n \forall x \in \mathcal{S}, n \in \mathbb{Z}^+$

Note that $A^0 = A^*$ see Gambo and Tella, (2022).

x. mset direct product

Let $A, B \in \mathfrak{M}(\mathcal{S})$. Then the direct product of A and B denoted by $A \otimes B$ is defined by $C_{A \otimes B}\left((x, y)\right) = \{(a/x, b/y)/a.b\colon\ x \in^a A, y \in^b B\}$.

xi. mset raising to the direct power

Let $A \in \mathfrak{M}(\mathcal{S})$. Then the mset A raising to the direct power of n denoted by $(\times A)^n$ is defined by $C_{(\times A)^n}(x_1, x_2, x_3, \ldots, x_n) = \prod_{i=1}^{n} C_A(x_i),\ x_i \in A$.

Multiset Inventory Operations

For multisets $M_1, M_2 \in \mathfrak{M}(\mathcal{S})$

1. Addition (Restocking):
$$C_{M_1+M_2}(x) = C_{M_1}(x) + C_{M_2}(x)$$
2. Difference (Sales/Usage)
$$C_{M_1-M_2}(x) = C_{M_1}(x) - C_{M_2}(x)$$
3. Union:
$$C_{M_1 \cup M_2}(x) = \max\left(C_{M_1}(x), C_{M_2}(x)\right)$$
4. Intersection:
$$C_{M_1 \cap M_2}(x) = \min\left(C_{M_1}(x), C_{M_2}(x)\right)$$

These operations reflect practical inventory process, including restocking, sales deduction and stock comparison Blizard, (1989), Zhou et al., (2020), Tella and Daniel, (2013)

Dynamic Inventory Model

Let M_t =inventory at time t

M_1 = restocking multiset

Discrete-time update:

$$M_{t+1} = M_t + R_t - S_t$$

Subject to:

$C_{M_t+1}(x) \geq 0$ for all $x \in \mathcal{S}$

Ensure feasibility and stability of the inventory system by Knuth, (1981), and Rahman & Yadav, (2022).

Algebraic Properties

Theorem (Closure)

$\mathfrak{M}(\mathcal{S})$ is closed under addition

Proof

Multiplicities are natural numbers; hence sum remains in $\mathbb{N}$. Blizard, (1989).

Theorem (Inventory stability)

If $S_t \subseteq M_t$ and M_0 is feasible, then M_t remains feasible for all t . Zhou et al., (2020).

Optimization Formulation

Let $D(x)$ denote the forecast demand. Define imbalance functional:

$J(M) = \sum_{x \in \mathcal{S}} |C_M(x) - D(x)|$

(1)

Optimization problem:

$\min_{m \in \mathfrak{M}(\mathcal{S})} J(M)$

Minimization discrepancy between inventory and demand, ensuring efficient allocation.

See Simchi-Levi et al., (2008). and Zhou et al., (2020).

Interpretation

The objective function minimizes the total deviation between available inventory and forecast demand. A smaller value $J(M)$ indicates better alignment between stock availability and customer requirements.

In motor garage operations, excessive inventory increases storage costs, while insufficient inventory causes service delays and customer dissatisfaction. Therefore the proposed optimization framework seeks a balance between these competing objectives by maintaining inventory levels closed to anticipated demand

RESULTS AND DISCUSSION

Computational Algorithm

Algorithm: Multiset Inventory Update

Input: M_0, R_t , S_t

For each $x \in X$:

$C(x) = C_{M_0}(x) - C_{R_t}(x) - C_{S_t}(x)$

If negative, set $C(x) = 0$

Output: Update inventory M_{t+1}

Complexity: $O(n)$, linear in inventory size. Knuth, (1981). and Rahman & Yadav, (2022).

Interpretation

Linear complexity makes the proposed framework suitable for small and medium-sized motor garages where inventory updates occur frequently. The algorithm remains computationally efficient even when the number of inventory items increases.

Case Study

Suppose:

$M_0 = \{plug(20), filter(15), bolt(50)\}$,

$R_t = \{plug(10), filter(5),\}$,
$S_t = \{plug(8), bolt(12)\}$,
Then:
$M_{t+1} = \{plug(22), filter(20), bolt(38)\}$,

Interpretation

The above case study demonstrates how multiset operations naturally model inventory transitions without requiring separate quantity variables. This illustrates the practicality of the proposed framework for day-to-day garage inventory management.

The results demonstrate that multiset theory provides an effective framework for modelling motor garage inventories containing repeated spare parts. By representing stock quantities through multiplicity functions, the proposed approach integrates item identity and quantity within a single mathematical structure, thereby overcoming a major limitation of classical set-based representations.

The closure and stability properties established in this study ensure that inventory operations such as restocking and sale deductions preserve the validity of the inventory system. Consequently, inventory updates can be performed consistently without generating infeasible stock levels.

The optimization formulation provides a mechanism for aligning available inventory with forecast demand. This assists motor garage operators in reducing stock shortages and avoiding excessive inventory accumulation, thereby supporting more efficient inventory control.

Furthermore, the proposed update algorism has linear computational complexity, making it suitable for small and medium-scale motor garage operations where inventory records require frequent updating. The case study illustrates how multiset operations can be applied to practical inventory management problems in a simple and computationally efficient manner.

Overall, the findings suggest that multiset algebra offers a mathematically rigorous and practically useful approach to inventory optimization in motor garage systems.

CONCLUSION

This study developed a multiset-theoretic framework for inventory optimization in motor garage systems. The framework models inventory items through multiplicity functions and formalizes inventory operations using multiset algebra. The results show that the proposed approach provides mathematically consistent and computationally efficient method for representing and updating inventory records. The optimization model supports improved alignment between available stock and demand, thereby enhancing inventory management decisions.

CONFLICT OF INTEREST

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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