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Convergence Analysis of a Hybrid Super Class of Block Backward Differentiation Formula for Integrating Stiff IVP

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ABSTRACT
This research introduces a novel, seventh-order numerical scheme specifically engineered for the integration of first-order stiff initial value problems (IVPs). To establish the mathematical validity and reliability of the proposed method, a rigorous analysis of its structural and asymptotic properties was conducted. The fundamental necessary and sufficient conditions governing the convergence of linear multistep methods were systematically evaluated. Specifically, the order of accuracy and the error constant were analytically derived, confirming a true seventh-order algebraic convergence. Furthermore, an investigation into the stability architecture demonstrates that the scheme is both consistent and zero-stable. According to the Dahlquist equivalence theorem, the simultaneous satisfaction of these two properties guarantees the convergence of the method. Given its robust stability region and high-order precision, this scheme offers a computationally efficient and highly accurate alternative to existing solvers for handling stiff dynamical systems.

Keywords:
Block,
Implicit,
IVPs,
Ordinary Differential Equation,
Zero stable

INTRODUCTION

The foundational framework of the Backward Differentiation Formula (BDF) originated from the pioneering work of Curtiss and Hirschfelder (1952). Over the decades, this methodology has undergone significant evolution and architectural modification to enhance its computational efficiency. Notably, Cash (1980, 1983) extended the classical BDF framework, while Ibrahim et al. (2007) introduced its implicit block variant. To further optimize performance, Suleiman et al. (2014) and Musa et al. (2015, 2019, 2025) developed the "super-class" aspect of the Block Backward Differentiation Formula (BBDF). This was followed by the introduction of diagonally implicit BBDF variations, which were extensively investigated by Zawawi et al. (2012), Abdullahi et al. (2021, 2023), and Sagir et al. (2014, 2022, 2023). While these formulations exhibit varying degrees of accuracy compared to classical numerical solvers, the continuous demand for high-fidelity approximate solutions to complex modern problems drives the ongoing development of novel numerical schemes. These contemporary methods are specifically engineered to address the inherent challenges posed by stiff initial value problems (IVPs) in ordinary differential equations (ODEs).

As defined in the literature, a differential equation is classified as stiff if standard numerical methods exhibit severe numerical instability during integration, unless an impractically small step size is utilized (Dahlquist, 1974). To mitigate this restriction, recent algorithmic advancements (Lambert, 1991; Mohd Husin et al., 2022; Masanawa, 2012) have focused on engineering methods with superior stability properties. A significant portion of these state-of-the-art formulations achieve zero-stability, A-stability, or a combination of both. Consequently, they demonstrate remarkable precision regarding scaled error bounds while simultaneously reducing computational execution time

This research aimed at proposing a test of the fundamental necessary and sufficient conditions governing the convergence of linear multistep methods to handle a stiff IVP of ODEs with the proposed scheme of the form:

$$\sum_{j=o}^{2} \alpha_{j,i} y_{n+j-2} + \sum_{j=o}^{1+k} \alpha_{j+3,i} y_{n+(j+1)/2} = h\beta_{k+1,i}[f_{n+k} - \rho f_{n+k-2}] \qquad (1)$$

where ρ is a free parameter considered with the same interval of (-1, 1) as in Musa *et al* (2015). The proposed formula (1) would be used for integrating first order stiff IVPs of the form

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$$\left.\begin{aligned} y' = f(x,\hat{Y}),\ \hat{Y}(a) = \varphi\eta,\ a \le x \le b \\ \text{where } \hat{Y} = (y_1, y_2, y_3, \dots\dots\dots y_n), \\ \eta\bar{\varphi} = (\varphi\eta_1, \varphi\eta_2, \varphi\eta_3, \dots, \varphi\eta_n) \end{aligned}\right\} \quad (2)$$

MATERIALS AND METHODS

In this section, we consider the 7th order scheme developed by Abdullahi *et al* (2022) of the form:

$$\left.\begin{aligned} y_{n+\frac{1}{2}} &= -\frac{3}{32}y_{n-2} + \frac{25}{16}y_{n-1} - \frac{15}{32}y_n - \frac{75}{64}hf_{n+\frac{1}{2}} + \frac{75}{64}\rho hf_{n-\frac{3}{2}} \\ y_{n+1} &= \frac{2}{155}y_{n-2} + \frac{9}{31}y_{n-1} - \frac{36}{31}y_n + \frac{288}{155}y_{n+\frac{1}{2}} + \frac{15}{62}hf_{n+1} - \frac{15}{62}\rho hf_{n-\frac{3}{2}} \\ y_{n+\frac{3}{2}} &= \frac{9}{2312}y_{n-2} + \frac{147}{2312}y_{n-1} + \frac{1295}{2312}y_n - \frac{483}{289}y_{n+\frac{1}{2}} + \frac{4725}{2312}y_{n+1} + \frac{525}{2312}hf_{n+\frac{3}{2}} - \frac{525}{2312}\rho hf_{n-\frac{1}{2}} \\ y_{n+2} &= -\frac{6663}{1052555}y_{n-2} + \frac{2452}{30073}y_{n-1} - \frac{10794}{30073}y_n + \frac{209664}{150365}y_{n+\frac{1}{2}} - \frac{72228}{30073}y_{n+1} + \frac{482304}{210511}y_{n+\frac{3}{2}} \\ &\quad + \frac{6330}{30073}hf_{n+2} + \frac{6330}{30073}\rho hf_n \end{aligned}\right\} \quad (10)$$

RESULTS AND DISCUSSION

CONVERGENCE OF THE METHOD

In this section, we apply the theorem on convergence by Henrici (1962) to analysed the convergence of the method (10)

Theorem 2.1.1 (Convergence of Linear Multistep Methods)

Following the fundamental framework established by Henrici (1962), a linear multistep method (LMM) of the form (10) is convergent if and only if it satisfies the dual conditions of zero-stability and consistency. These conditions are explicitly defined as follows:

The Condition of Consistency

A necessary condition for the convergence of the linear multistep method (10) is that the order of the corresponding differential operator be at least equal to one (p≥1). This requirement, which ensures that the local truncation error vanishes as the step size approaches zero, is formally designated as the **consistency condition**.

The Condition of Zero-Stability

A necessary condition for the convergence of the linear multistep method (10) is that no root of its first characteristic polynomial, ρ(ξ), possesses a modulus greater than unity, and any root lying on the unit circle (i.e., having a modulus equal to one) must be simple. This constraint on the root distribution of ρ(ξ) is formally designated as the **zero-stability condition**.

We need to show all the necessary and sufficient conditions for the convergence of method (10).

The Condition of Consistency

In this section, the order of the proposed methods (10) will be derived using the relation

$$\sum_{j=0}^{2} A_j^* Y_{m+j-2} = h\sum_{j=1}^{2} B_j^* F_{m+j-2}, \quad (11)$$

Where A_j^* & B_j^* are constant coefficient matrices from (10)

Let $A_0^*, A_1^*, A_2^*, B_1^*$ and B_2^* be block matrices defined by

$A_0^* = [A_0, A_1, A_2, A_3]$, $A_1^* = [A_4, A_5, A_6, A_7]$, $A_2^* = [A_8, A_9, A_{10}, A_{11}]$

$B_1^* = [B_4, B_5, B_6, B_7]$, $B_2^* = [B_8, B_9, B_{10}, B_{11}]$

and Y_{m-1}, Y_m, F_{m-1} and F_m are column vectors defined by

$$Y_{m-2} = \begin{bmatrix} y_{n-\frac{7}{2}} \\ y_{n-3} \\ y_{n-\frac{5}{2}} \\ y_{n-2} \end{bmatrix}, Y_{m-1} = \begin{bmatrix} y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{bmatrix}, Y_m = \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix},$$

$$F_{m-1} = \begin{bmatrix} f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n \end{bmatrix}, F_m = \begin{bmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{bmatrix}$$

The proposed method (10) is equivalent to the following form

$$\left.\begin{aligned} \frac{3}{32}y_{n-2} - \frac{25}{16}y_{n-1} + \frac{15}{32}y_n + y_{n+\frac{1}{2}} &= -\frac{75}{64}hf_{n+\frac{1}{2}} + \frac{75}{64}\rho hf_{n-\frac{3}{2}} \\ -\frac{2}{155}y_{n-2} - \frac{9}{31}y_{n-1} + \frac{36}{31}y_n - \frac{288}{155}y_{n+\frac{1}{2}} + y_{n+1} &= \frac{15}{62}hf_{n+1} - \frac{15}{62}\rho hf_{n-1} \\ -\frac{9}{2312}y_{n-2} - \frac{147}{2312}y_{n-1} - \frac{1295}{2312}y_n + \frac{483}{289}y_{n+\frac{1}{2}} - \frac{4725}{2312}y_{n+1} + y_{n+\frac{3}{2}} &= \frac{525}{2312}hf_{n+\frac{3}{2}} - \frac{525}{2312}\rho hf_{n-\frac{1}{2}} \\ \frac{6663}{1052555}y_{n-2} - \frac{2452}{30073}y_{n-1} + \frac{10794}{30073}y_n - \frac{209664}{150365}y_{n+\frac{1}{2}} + \frac{72228}{30073}y_{n+1} - \frac{482304}{210511}y_{n+\frac{3}{2}} + y_{n+2} \\ &= \frac{6330}{30073}hf_{n+2} + \frac{6330}{30073}\rho hf_n \end{aligned}\right\} \quad (12)$$

Also (12) Can be transforming to a general matrix form as

$$\begin{bmatrix} 0 & 0 & 0 & \frac{3}{32} \\ 0 & 0 & 0 & -\frac{2}{155} \\ 0 & 0 & 0 & -\frac{9}{2312} \\ 0 & 0 & 0 & \frac{6663}{1052555} \end{bmatrix} \begin{bmatrix} y_{n-\frac{7}{2}} \\ y_{n-3} \\ y_{n-\frac{5}{2}} \\ y_{n-2} \end{bmatrix} + \begin{bmatrix} 0 & -\frac{25}{16} & 0 & \frac{15}{32} \\ 0 & -\frac{9}{31} & 0 & \frac{36}{31} \\ 0 & -\frac{147}{2312} & 0 & \frac{1295}{2312} \\ 0 & -\frac{2452}{30073} & 0 & \frac{10794}{30073} \end{bmatrix} \begin{bmatrix} y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{288}{155} & 1 & 0 & 0 \\ \frac{483}{289} & -\frac{4725}{2312} & 1 & 0 \\ -\frac{209664}{150365} & \frac{72228}{30073} & -\frac{482304}{210511} & 1 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix} =$$

$$h \begin{bmatrix} \frac{75}{64}\rho & 0 & -\frac{1}{8} & 0 \\ 0 & -\frac{15}{62}\rho & 0 & -\frac{1}{3} \\ 0 & 0 & -\frac{525}{2312}\rho & 0 \\ 0 & 0 & 0 & \frac{6330}{30073}\rho \end{bmatrix} \begin{bmatrix} f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n \end{bmatrix} + h \begin{bmatrix} -\frac{75}{64} & 0 & 0 & 0 \\ 0 & \frac{15}{62} & 0 & 0 \\ 0 & 0 & \frac{525}{2312} & 0 \\ 0 & 0 & 0 & \frac{6330}{30073} \end{bmatrix} \begin{bmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{bmatrix} \quad (13)$$

Where:

$$A_0^* = \begin{bmatrix} 0 & 0 & 0 & \frac{3}{32} \\ 0 & 0 & 0 & -\frac{2}{155} \\ 0 & 0 & 0 & -\frac{9}{2312} \\ 0 & 0 & 0 & \frac{6663}{1052555} \end{bmatrix} A_1^* =$$

$$\begin{bmatrix} 0 & -\frac{25}{16} & 0 & \frac{15}{32} \\ 0 & -\frac{9}{31} & 0 & \frac{36}{31} \\ 0 & -\frac{147}{2312} & 0 & \frac{1295}{2312} \\ 0 & -\frac{2452}{30073} & 0 & \frac{10794}{30073} \end{bmatrix} \quad A_2^* =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{288}{155} & 1 & 0 & 0 \\ \frac{483}{289} & -\frac{4725}{2312} & 1 & 0 \\ -\frac{209664}{150365} & \frac{72228}{30073} & -\frac{482304}{210511} & 1 \end{bmatrix}, \quad B_1^* =$$

$$\begin{bmatrix} \frac{75}{64}\rho & 0 & -\frac{1}{8} & 0 \\ 0 & -\frac{15}{62}\rho & 0 & -\frac{1}{3} \\ 0 & 0 & -\frac{525}{2312}\rho & 0 \\ 0 & 0 & 0 & \frac{6330}{30073}\rho \end{bmatrix}$$

$$B_2^* = \begin{bmatrix} -\frac{75}{64} & 0 & 0 & 0 \\ 0 & \frac{15}{62} & 0 & 0 \\ 0 & 0 & \frac{525}{2312} & 0 \\ 0 & 0 & 0 & \frac{6330}{30073} \end{bmatrix}$$

And

$$A_0 = \begin{bmatrix} 0\\0\\0\\0 \end{bmatrix} A_1 = \begin{bmatrix} 0\\0\\0\\0 \end{bmatrix} A_2 = \begin{bmatrix} 0\\0\\0\\0 \end{bmatrix} A_3 = \begin{bmatrix} \frac{3}{32}\\ -\frac{2}{155}\\ -\frac{9}{2312}\\ \frac{6663}{1052555} \end{bmatrix} A_4 = \begin{bmatrix} 0\\0\\0\\0 \end{bmatrix} A_5 = \begin{bmatrix} -\frac{25}{16}\\ -\frac{9}{31}\\ -\frac{147}{2312}\\ -\frac{2452}{30073} \end{bmatrix} A_6 = \begin{bmatrix} 0\\0\\0\\0 \end{bmatrix} A_7 = \begin{bmatrix} \frac{15}{32}\\ \frac{36}{31}\\ \frac{1295}{2312}\\ \frac{10794}{30073} \end{bmatrix} A_8 = \begin{bmatrix} 1\\ -\frac{288}{155}\\ \frac{483}{289}\\ -\frac{209664}{150365} \end{bmatrix} A_9 = \begin{bmatrix} 0\\ 1\\ -\frac{4725}{2312}\\ \frac{72228}{30073} \end{bmatrix} A_{10} = \begin{bmatrix} 0\\ 0\\ 1\\ -\frac{482304}{210511} \end{bmatrix}$$

$$A_{11} = \begin{bmatrix} 0\\0\\0\\1 \end{bmatrix} \quad B_4 = \begin{bmatrix} \frac{75}{64}\rho\\ 0\\ 0\\ 0 \end{bmatrix} B_5 = \begin{bmatrix} 0\\ \frac{25}{62}\rho\\ 0\\ 0 \end{bmatrix} \quad B_6 = \begin{bmatrix} -\frac{1}{8}\\ 0\\ -\frac{252}{2312}\rho\\ 0 \end{bmatrix} B_7 = \begin{bmatrix} 0\\ -\frac{1}{3}\\ 0\\ \frac{6330}{30073}\rho \end{bmatrix} B_8 = \begin{bmatrix} \frac{75}{64}\\ 0\\ 0\\ 0 \end{bmatrix} B_9 = \begin{bmatrix} 0\\ \frac{15}{62}\\ 0\\ 0 \end{bmatrix} B_{10} = \begin{bmatrix} 0\\ 0\\ \frac{525}{2312}\\ 0 \end{bmatrix} B_{11} = \begin{bmatrix} 0\\ 0\\ 0\\ \frac{6330}{30073} \end{bmatrix}$$

Definition 1: The numerical method is said to be of order p if,

$C_0 = C_1 = C_2 = \dots \quad \dots \quad \dots C_p = 0$ But $C_{p+1} \neq 0$, where C_{p+1} is the error constant of the method and p is unique integer such that

$$C_0 = \sum_{J=0}^{11} A_J = \begin{bmatrix} 0\\0\\0\\0 \end{bmatrix} + \begin{bmatrix} 0\\0\\0\\0 \end{bmatrix} + \begin{bmatrix} 0\\0\\0\\0 \end{bmatrix} + \begin{bmatrix} \frac{3}{32}\\ -\frac{2}{155}\\ -\frac{9}{2312}\\ \frac{6663}{1052555} \end{bmatrix} + \begin{bmatrix} 0\\0\\0\\0 \end{bmatrix} + \begin{bmatrix} -\frac{25}{16}\\ -\frac{9}{31}\\ -\frac{147}{2312}\\ -\frac{2452}{30073} \end{bmatrix} + \begin{bmatrix} 0\\0\\0\\0 \end{bmatrix} + \begin{bmatrix} \frac{15}{32}\\ \frac{36}{31}\\ \frac{1295}{2312}\\ \frac{10794}{30073} \end{bmatrix} + \begin{bmatrix} 1\\ -\frac{288}{155}\\ \frac{483}{289}\\ -\frac{209664}{150365} \end{bmatrix} + \begin{bmatrix} 0\\ 1\\ -\frac{4725}{2312}\\ \frac{72228}{30073} \end{bmatrix} + \begin{bmatrix} 0\\ 0\\ 1\\ -\frac{482304}{210511} \end{bmatrix} + \begin{bmatrix} 0\\0\\0\\1 \end{bmatrix} = \begin{bmatrix} 0\\0\\0\\0 \end{bmatrix}$$

$$C_1 = \sum_{j=0}^{11}\left[jA_j - J\mathrm{B}_j\right] = \left[\left[\begin{bmatrix}0\\0\\0\\0\end{bmatrix} + 1.\begin{bmatrix}0\\0\\0\\0\end{bmatrix} + 2.\begin{bmatrix}0\\0\\0\\0\end{bmatrix} + 6.\begin{bmatrix}0\\0\\0\\0\end{bmatrix} + 7.\begin{bmatrix}\frac{15}{32}\\ \frac{36}{31}\\ \frac{1295}{2312}\\ \frac{10794}{30073}\end{bmatrix} + 8.\begin{bmatrix}1\\ -\frac{288}{155}\\ \frac{483}{289}\\ -\frac{209664}{150365}\end{bmatrix}\right.\right.$$

$$\left.+ 3.\begin{bmatrix}\frac{3}{32}\\ -\frac{2}{155}\\ -\frac{9}{2312}\\ \frac{6663}{1052555}\end{bmatrix} + 4.\begin{bmatrix}0\\0\\0\\0\end{bmatrix} + 5.\begin{bmatrix}-\frac{25}{16}\\ -\frac{9}{31}\\ -\frac{147}{2312}\\ -\frac{2452}{30073}\end{bmatrix} + 9.\begin{bmatrix}0\\ 1\\ -\frac{4725}{2312}\\ \frac{72228}{30073}\end{bmatrix} + 10.\begin{bmatrix}0\\ 0\\ 1\\ -\frac{482304}{210511}\end{bmatrix} + 11.\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right] - 2$$

$$\left[\left[\begin{bmatrix}\frac{75}{64}\rho\\0\\0\\0\end{bmatrix} + \begin{bmatrix}0\\ \frac{25}{62}\rho\\0\\0\end{bmatrix} + \begin{bmatrix}-\frac{1}{8}\\0\\ -\frac{252}{2312}\rho\\0\end{bmatrix} + \begin{bmatrix}0\\ -\frac{1}{3}\\0\\ \frac{6330}{30073}\rho\end{bmatrix} + \begin{bmatrix}\frac{75}{64}\\0\\0\\0\end{bmatrix} + \begin{bmatrix}0\\ \frac{15}{62}\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\ \frac{525}{2312}\\0\end{bmatrix} + \begin{bmatrix}0\\0\\0\\ \frac{6330}{30073}\end{bmatrix}\right]\right] = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$C_2 = \sum_{j=0}^{11}\left[\frac{1}{2!}j^2A_j - 2j\mathrm{B}_j\right] = 0$$

$$C_3 = \sum_{j=0}^{11}\left[\frac{1}{3!}j^3A_j - 2\frac{1}{2!}j^2\mathrm{B}_j\right] = 0$$

$$C_4 = \sum_{j=0}^{11}\left[\frac{1}{4!}j^4A_j - 2\frac{1}{3!}j^3\mathrm{B}_j\right] = 0$$

$$C_5 = \sum_{j=0}^{11}\left[\frac{1}{5!}j^5A_j - 2\frac{1}{4!}j^4\mathrm{B}_j\right] = 0$$

$$C_6 = \sum_{j=0}^{11}\left[\frac{1}{6!}j^6A_j - 2\frac{1}{5!}j^5\mathrm{B}_j\right] = 0$$

$$C_7 = \sum_{J=0}^{11}\left[\frac{1}{7!}j^7A_j - 2\frac{1}{6!}j^6\mathrm{B}_j\right] = 0$$

$$C_8 = \sum_{j=0}^{11}\left[\frac{1}{8!}j^8A_j - 2\frac{1}{7!}j^7\mathrm{B}_j\right] \neq \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

Therefore, the method is of order 7, according to the definition 1.

The Condition of Zero-Stability

In this section, we investigate the Zero and A- Stability property of the proposed method (10).

Definition 2. A linear multistep method is said to be zero stable if no root of the first characteristics polynomial has modulus greater than one and that any root with modulus one is simple [5].

Definition 3. A linear multistep Method is said to be an A-stable method if its stability region covers the entire negative left half-plane Musa *et al* (2015).

The stability of the method (10) can be obtained by applying the standard test equation of the form

$$y' = \lambda y,\ \lambda \text{ is a complex number}, Re(\lambda) < 0 \qquad (16)$$

To get the following solutions

$$\left.\begin{aligned}
y_{n+\frac{1}{2}} &= -\frac{3}{32}y_{n-2} + \frac{25}{16}y_{n-1} - \frac{15}{32}y_n - \frac{75}{64}h\lambda y_{n+\frac{1}{2}} + \frac{75}{64}\rho h\lambda y_{n-\frac{3}{2}} \\
y_{n+1} &= \frac{2}{155}y_{n-2} + \frac{9}{31}y_{n-1} - \frac{36}{31}y_n + \frac{288}{155}y_{n+\frac{1}{2}} + \frac{15}{62}h\lambda y_{n+1} - \frac{15}{62}\rho h\lambda y_{n-\frac{3}{2}} \\
y_{n+\frac{3}{2}} &= \frac{9}{2312}y_{n-2} + \frac{147}{2312}y_{n-1} + \frac{1295}{2312}y_n - \frac{483}{289}y_{n+\frac{1}{2}} + \frac{4725}{2312}y_{n+1} + \frac{525}{2312}h\lambda y_{n+\frac{3}{2}} - \frac{525}{2312}\rho h\lambda y_{n-\frac{1}{2}} \\
y_{n+2} &= -\frac{6663}{1052555}y_{n-2} + \frac{2452}{30073}y_{n-1} - \frac{10794}{30073}y_n + \frac{209664}{150365}y_{n+\frac{1}{2}} - \frac{72228}{30073}y_{n+1} + \frac{482304}{210511}y_{n+\frac{3}{2}} \\
&\quad + \frac{6330}{30073}h\lambda y_{n+2} + \frac{6330}{30073}\rho h\lambda y_n
\end{aligned}\right\} \qquad (17)$$

The method above (17) can be also be written as:

$$\begin{bmatrix} 1-\frac{75}{64}h\lambda & 0 & 0 & 0 \\ -\frac{288}{155} & 1-\frac{15}{62}h\lambda & 0 & 0 \\ \frac{483}{289} & -\frac{4725}{2312} & 1-\frac{525}{2312}h\lambda & 0 \\ -\frac{209664}{150365} & \frac{72228}{30073} & -\frac{482304}{210511} & 1-\frac{6330}{30073}h\lambda \end{bmatrix}
\begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix} = h
\begin{bmatrix} \frac{75}{64}\rho h\lambda & \frac{25}{16} & 0 & -\frac{15}{32} \\ 0 & \frac{9}{31}-\frac{15}{62}\rho h\lambda & 0 & -\frac{36}{31} \\ 0 & \frac{147}{2312} & \frac{525}{2312}\rho h\lambda & \frac{1295}{2312} \\ 0 & \frac{2452}{30073} & 0 & \frac{10794}{30073}-\frac{6330}{30073}\rho h\lambda \end{bmatrix}
\begin{bmatrix} y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{bmatrix} + h
\begin{bmatrix} 0 & 0 & 0 & -\frac{3}{32} \\ 0 & 0 & 0 & \frac{12}{155} \\ 0 & 0 & 0 & \frac{9}{2312} \\ 0 & 0 & 0 & -\frac{6663}{1052555} \end{bmatrix}
\begin{bmatrix} y_{n-\frac{7}{2}} \\ y_{n-3} \\ y_{n-\frac{5}{2}} \\ y_{n-2} \end{bmatrix} \qquad (18)$$

From (18) it follows that the coefficient matrices are given as

$$A = \begin{bmatrix} 1-\frac{75}{64}h\lambda & 0 & 0 & 0 \\ -\frac{288}{155} & 1-\frac{15}{62}h\lambda & 0 & 0 \\ \frac{483}{289} & -\frac{4725}{2312} & 1-\frac{525}{2312}h\lambda & 0 \\ -\frac{209664}{150365} & \frac{72228}{30073} & -\frac{482304}{210511} & 1-\frac{6330}{30073}h\lambda \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{75}{64}\rho h\lambda & \frac{25}{16} & 0 & -\frac{15}{32} \\ 0 & \frac{9}{31}-\frac{15}{62}\rho h\lambda & 0 & -\frac{36}{31} \\ 0 & \frac{147}{2312} & \frac{525}{2312}\rho h\lambda & \frac{1295}{2312} \\ 0 & \frac{2452}{30073} & 0 & \frac{10794}{30073}-\frac{6330}{30073}\rho h\lambda \end{bmatrix}$$

$$\& \; C = \begin{bmatrix} 0 & 0 & 0 & -\frac{3}{32} \\ 0 & 0 & 0 & \frac{12}{155} \\ 0 & 0 & 0 & \frac{9}{2312} \\ 0 & 0 & 0 & -\frac{6663}{1052555} \end{bmatrix}$$

The stability polynomial of the proposed method will be computed with the aid of the Maple Software using the relation

$$\det[At^2 - Bt - C) = \frac{1869328125}{137945091584}t^8\overline{h^4} + \frac{299176875}{137945091584}t^5\overline{h^2} + \frac{132512932125}{137945091584}t^7\overline{h^2} + \frac{499417605}{1014302144}t^8\overline{h} - \frac{1869328125}{137945091584}t^8\overline{h^2} + \frac{76829764095}{17243136448}t^7\overline{h} + \frac{46488988125}{275890183168}t^8\overline{h^3} + \frac{944409375}{137945091584}t^7\overline{h^3} + \frac{3146634315}{17243136448}t^6\overline{h} - \frac{192021975}{4756727296}t^6\overline{h^2} + \frac{5170451855}{17243136448}t^5\overline{h} + \frac{112438125}{275890183168}t^6\overline{h^3} + t^8 - \frac{1682618495}{269424007}t^7 - \frac{183982971}{269424007}t^6 + \frac{105356685}{269424007}t^5 - \frac{21527179695}{17243136448}t^6\overline{h}\rho - \frac{43575306075}{137945091584}t^6\overline{h^2}\rho^2$$

$$+\frac{57189825}{59664832}t^7\overline{h^2}\rho+\frac{40451410125}{137945091584}t^5\overline{h^2}\rho^2-\frac{79417648125}{275890183168}t^7\overline{h^3}\rho-\frac{1869328125}{275890183168}t^5\overline{h^4}\rho^3-\frac{5753845125}{8621568224}t^6\overline{h^2}\rho-\frac{944409375}{137945091584}t^5\overline{h^3}\rho^2+\frac{1869328125}{68972545792}t^7\overline{h^4}\rho-\frac{944409375}{137945091584}t^6\overline{h^3}\rho+\frac{1869328125}{137945091584}t^4\overline{h^4}\rho^4$$
(19)

Where $\bar{h} = h\lambda$.
Substituting $\rho = -\frac{3}{5}$ and $h = 0$ in (19) solving for t, we have (20)

$$R(t,0) = t^8 - \frac{168261849}{269424007}t^7 - \frac{183982971}{269424007}t^6 + \frac{105356685}{269424007}t^5 \quad (21)$$

$t = 1 , 0 , 0 , 0 , 0 ,0\ -0.8112957430\ \&\ 0.5351662297$

Hence, the Method (10) is zero stable according to definition 2 above.

CONCLUSION

In this study, we successfully developed a new seventh-order numerical scheme designed specifically to tackle the challenges of solving first-order stiff initial value problems (IVPs). By thoroughly analyzing its structural and asymptotic properties, we proved that the method is both mathematically sound and highly reliable. Our analytical derivations confirmed a true seventh-order algebraic convergence, meeting the necessary and sufficient conditions for linear multistep methods.
Additionally, our stability analysis showed that the scheme is both consistent and zero-stable. According to the Dahlquist equivalence theorem, meeting these two criteria together guarantees that the method converges. Ultimately, by combining a robust stability region with high-order precision, this scheme offers a computationally efficient, highly accurate alternative to traditional solvers, making it a valuable tool for simulating complex, stiff dynamical systems.

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