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On the Numerical Instability of the Riemann–Liouville Fractional Derivative: Analysis of Errors and Convergence Behaviour

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ABSTRACT

This paper develops a second-order numerical approximation for the Riemann–Liouville fractional derivative using a modified product trapezoidal rule. The method is derived from the fractional integral formulation via piecewise linear interpolation, yielding a discrete convolution depending only on function values. A rigorous error analysis establishes $O(h^2)$ accuracy under smoothness assumptions. However, numerical experiments on a test problem reveal significant discrepancies between theory and computation. The results exhibit instability, rapid error growth, and negative convergence orders for fractional orders $\nu \in (0,2]$. The divergence intensifies as the mesh is refined, indicating a breakdown of numerical consistency. These findings reveal notable discrepancies between theoretical predictions and computational performance, highlighting inherent limitations of the Riemann–Liouville derivative operator in applications.

Keywords:
Riemann–Liouville
Fractional derivative;
Product trapezoidal rule;
Convolution quadrature;
Second-order scheme;
Error analysis;
Weakly singular kernels

INTRODUCTION

Fractional calculus is a generalisation of classical calculus that extends the concepts of differentiation and integration from integer orders to arbitrary (non-integer) orders. Despite its name, it does not refer to a "fraction of calculus," but rather to calculus of fractional order operators. The origins of fractional calculus date back to 1695, in a correspondence between Leibniz and L'Hôpital, and the subject has since evolved into a well-established field of mathematical analysis (Miller and Ross, 1993; Oldham and Spanier, 1974; Samko *et al.*, 1993; Podlubny, 1999).

In recent decades, fractional calculus has attracted considerable attention due to its ability to model complex phenomena more accurately than classical integer-order models. In particular, fractional derivatives provide an effective framework for describing memory and hereditary properties inherent in many physical, biological, and engineering systems. As a result, they have found widespread applications in areas such as viscoelasticity, anomalous diffusion, control theory, signal processing, finance, and bioengineering (Poddubny, 1999; Bagley and Torvik, 1983; Metzler and Klafter, 2004; Sun *et al.*, 2018; Tarasov, 2013; Magin, 2006; Hilfer, 2000; Petráš, 2011).

As a practical application to real-world problems, Agbata *et al.* (2025) developed a novel fractional-order mathematical model for investigating the transmission dynamics of typhoid fever. Their model incorporates memory effects and complex transmission mechanisms that conventional integer-order models are unable to adequately capture. Comprehensive treatments of both the theory and applications of fractional calculus can be found in standard monographs such as Miller and Ross (1993), Oldham and Spanier (1974), Samko *et al.* (1993), and Podlubny (1999).

A variety of analytical and numerical techniques have been developed for solving fractional differential equations arising in practical applications. Classical approaches include Laplace and Fourier transform methods (Miller and Ross, 1993), while numerical methods such as finite difference schemes (Meerschaert *et al.*, 2006) and fractional linear multistep methods (Lubich, 1985, Lubich, 1986) have been widely studied. In addition, several semi-analytical methods have been proposed, including the Adomian decomposition method, variational iteration method, homotopy perturbation method, and generalized differential transform method (Momani, 2006; Momani and Odibat, 2006; Odibat and Momani 2006; Odibat and Momani, 2008). These approaches have significantly contributed to the advancement of computational fractional calculus.

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Unlike classical derivatives, fractional derivatives are not uniquely defined. Among the various definitions available, the Riemann–Liouville (R–L) and Caputo derivatives are the most widely used in theory and applications. The Riemann–Liouville derivative possesses strong analytical properties and arises naturally from the fractional integral operator. However, it involves fractional-order initial conditions, which are often difficult to interpret in practical applications. On the other hand, the Caputo derivative allows the use of classical integer-order initial conditions, making it more suitable for modelling real-world problems, although it lacks certain analytical advantages of the Riemann–Liouville formulation.

While substantial progress has been made in the numerical approximation of Caputo fractional derivatives and fractional integrals (Odibat, 2006; Odibat, 2009; Aboiyar and Isah, 2015), comparatively less attention has been devoted to the direct and efficient approximation of the Riemann–Liouville fractional derivative. In particular, many existing approaches rely on transformations from Caputo formulations or require the evaluation of integer-order derivatives, which may increase computational complexity.

Motivated by these challenges, the primary objective of this paper is to develop a second-order accurate numerical scheme for the Riemann–Liouville fractional derivative based on a modified product trapezoidal rule. The proposed approach is constructed in a unified framework that depends only on function values, thereby avoiding the need for integer-order derivatives. In addition, a rigorous error analysis is provided to establish the convergence properties of the method, and extensive numerical experiments are conducted to validate its accuracy and stability across a range of fractional orders.

The remainder of this paper is organised as follows. In Section 2, we present the necessary preliminaries, including definitions of fractional operators and auxiliary results underlying the proposed approach. Section 3 is devoted to the derivation of a second-order numerical scheme for the Riemann–Liouville fractional derivative, together with a rigorous error and convergence analysis. In Section 4, numerical experiments are carried out to investigate the accuracy, stability, and convergence behaviour of the method across a range of fractional orders. The results reveal notable discrepancies between theoretical predictions and computational performance, highlighting inherent limitations of the Riemann–Liouville derivative operator in applications. Finally, Section 5 summarizes the main findings and discusses implications for practical applications, along with possible directions for future research.

MATERIALS AND METHODS

In this section, we present the necessary definitions and theorems.

Composite Trapezoidal Rule

Definition 1 (Gerald and Wheatley, 2004; Mathews and Fink, 1999). Let $[c,d]$ be divided into M equal subintervals with step size $h=\frac{d-c}{M}$ and nodes $x_i = c + ih$ for $i = 1,2,\dots,M$. The classical composite trapezoidal rule is defined as

$$C(f,h) = \frac{h}{2}\sum_{i=1}^{M}\big(f(x_{i-1}) + f(x_i)\big) \qquad (1)$$

or equivalently

$$C(f,h) = \frac{h}{2}(f(c) + f(d)) + h\sum_{i=1}^{M-1} f(x_i) \qquad (2)$$

Thus,

$$\int_c^d f(x)\,dx \approx C(f,h). \qquad (3)$$

Error:

$$E(f,h) = -\frac{(d-c)}{12} f''(k)h^2 = O(h^2). \qquad (4)$$

Product Trapezoidal Approximation for Fractional Integrals

Theorem 1. (Diethelm *et al.*, 2004). Let $f \in C^2[0,T]$, and let the uniform grid be defined by

$$t_i = ih, h = \frac{T}{M}, i = 0,1,\dots,M.$$

Let $\tilde{f}_M$ be the piecewise linear interpolant of f on each subinterval $[t_{i,}\, t_{i+1}]$.

Then, for any $\alpha > 0$ and $t_M = Mh$, the following representation holds:

$$\int_0^{t_M} (t_M - t)^{\alpha-1}\tilde{f}_M(t)\,dt = \sum_{i=0}^{M} a_{i,M}\, f(t_i), \qquad (5)$$

where the weights $a_{i,M}$ are given by

$$a_{i,M} = \begin{cases} \dfrac{h^\alpha}{\alpha(\alpha+1)}[(M-1)^{\alpha+1} - (M-1-\alpha)M^\alpha], & i = 0, \\ \dfrac{h^\alpha}{\alpha(\alpha+1)}\begin{bmatrix}(M-i+1)^{\alpha+1} + \\ (M-i-1)^{\alpha+1} \\ -2(M-i)^{\alpha+1}\end{bmatrix}, & 1 \le i \le M-1, \\ \dfrac{h^\alpha}{\alpha(\alpha+1)}, & i = M. \end{cases} \qquad (6)$$

Moreover, the approximation error satisfies

$$\left| \int_0^{t_M} (t_M - t)^{\alpha-1} f(t)\,dt - \sum_{i=0}^{M} a_{i,M}\, f(t_i) \right| \le C_\alpha \| f'' \|_\infty\ t_M^\alpha h^2, \qquad (7)$$

where C_α is a constant depending only on α.

Proof.

Step 1: Piecewise linear interpolation.

On each subinterval $[t_i, t_{i+1}]$, define

$$\tilde{f}_M(t) = \frac{t_{i+1} - t}{h} f(t_i) + \frac{t - t_i}{h} f(t_{i+1}), t \in [t_i, t_{i+1}].$$

Step 2: Decomposition of the integral.

We write

$$\int_0^{t_M} (t_M - t)^{\alpha-1} \tilde{f}_M(t)\, dt$$

$$= \sum_{i=0}^{M-1} \int_{t_i}^{t_{i+1}} (t_M - t)^{\alpha-1} \tilde{f}_M(t)\, dt.$$

Substituting $\tilde{f}_M(t)$ yields

$$I = \sum_{i=0}^{M-1} \int_{t_i}^{t_{i+1}} (t_M - t)^{\alpha-1} \left[\begin{array}{l} \frac{t_{i+1} - t}{h} f(t_i) + \\ \frac{t - t_i}{h} f(t_{i+1}) \end{array}\right] dt \qquad (8)$$

$$= \sum_{i=0}^{M-1} \left[f(t_i) I_i^{(1)} + f(t_{i+1}) I_i^{(2)}\right], \qquad (9)$$

where

$$I_i^{(1)} = \frac{1}{h} \int_{t_i}^{t_{i+1}} (t_M - t)^{\alpha-1} (t_{i+1} - t)\, dt, \qquad (10)$$

$$I_i^{(2)} = \frac{1}{h} \int_{t_i}^{t_{i+1}} (t_M - t)^{\alpha-1} (t - t_i)\, dt. \qquad (11)$$

Step 3: Change of variables.

Let $s = t_M - t$, then $dt = -ds$. The limits transform as $t = t_i \Rightarrow s = (M - i)h, t = t_{i+1} \Rightarrow s = (M - i - 1)h.$

Thus, the integrals reduce to

$$I_i^{(1)}, \qquad I_i^{(2)} \sim \int s^{\alpha-1} (\cdot)\, ds,$$

which can be evaluated exactly using

$$\int s^{\alpha-1} ds = \frac{s^\alpha}{\alpha}, \qquad \int s^\alpha ds = \frac{s^{\alpha+1}}{\alpha + 1}.$$

Carrying out these computations yields the weights $a_{i,M}$ in (6), and hence

$$\int_0^{t_M} (t_M - t)^{\alpha-1} \tilde{f}_M(t)\, dt = \sum_{i=0}^{M} a_{i,M} f(t_i).$$

Step 4: Error analysis.

Let

$$E = \int_0^{t_M} (t_M - t)^{\alpha-1} \left(f(t) - \tilde{f}_M(t)\right) dt.$$

Using interpolation theory, for $t \in [t_i, t_{i+1}]$,

$$f(t) - \tilde{f}_M(t) = \frac{f''(\xi_t)}{2} (t - t_i)(t - t_{i+1}),$$

for some $\xi_t \in [t_i, t_{i+1}]$.

Hence,

$$\mid f(t) - \tilde{f}_M(t) \mid \leq \frac{\| f'' \|_\infty}{2} \mid (t - t_i)(t - t_{i+1}) \mid.$$

Since

$$\mid (t - t_i)(t - t_{i+1}) \mid \leq \frac{h^2}{4},$$

we obtain

$$\mid E \mid \leq \frac{\| f'' \|_\infty}{2} \sum_{i=0}^{M-1} \int_{t_i}^{t_{i+1}} (t_M - t)^{\alpha-1} \frac{h^2}{4} dt \qquad (12)$$

$$\leq C_\alpha \| f'' \|_\infty h^2 \int_0^{t_M} (t_M - t)^{\alpha-1} dt. \qquad (13)$$

Finally,

$$\int_0^{t_M} (t_M - t)^{\alpha-1} dt = \frac{t_M^\alpha}{\alpha},$$

which gives

$$\mid E \mid \leq C_\alpha \| f'' \|_\infty t_M^\alpha h^2.$$

This completes the proof.

Riemann–Liouville Fractional Integral and Derivative, Caputo Fractional Derivative

Definition 2 (Kai, 2010; Kilbas *et al.*, 2006; Miller and Ross, 1993). Let f be a suitable function and ν be any complex parameter, where $n - 1 < \nu \leq n$. Then, the Riemann–Liouville fractional integral and derivative of order ν with lower limit $0 \in \mathbb{R}$ are defined, respectively, by

$${}^{RL}_{0}D^{\nu}_{x} f(x) = \frac{1}{\Gamma(\nu)} \int_0^x (x - t)^{\nu-1} f(t)\, \mathrm{d}t, \mathrm{Re}(\nu) > 0 \qquad (14)$$

$${}^{RL}_{0}D^{\nu}_{x} f(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left({}^{RL}_{0}I^{n-\nu}_{x} f(x)\right), \mathrm{Re}(\nu) \geq 0 \qquad (15)$$

where $x > 0$ and $n \in \mathbb{N}$ satisfies $n - 1 \leq \mathrm{Re}(\nu) < n$.

Similar to the Riemann–Liouville fractional derivative is the Caputo fractional derivative defined as

$${}^{C}_{0}D^{\nu}_{x} f(x) = {}^{RL}_{0}I^{n-\nu}_{x} \left(\frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x)\right), \qquad \mathrm{Re}(\nu) \geq 0 \qquad (16)$$

where n is the smallest integer that exceeds ν.

Details and properties of the Riemann–Liouville fractional integral are found in (Oldham, 1975; Samko *et al.*, 1993; Oldham and Spanier, 1974).

A Fundamental Relation between Riemann–Liouville and Caputo Fractional Derivatives

Theorem 2. Let $f \in C^n[0,1]$ and let $n-1 < \nu \le n$, where $n \in \mathbb{N}$. Then the Caputo and Riemann–Liouville fractional derivatives of order ν are related by the relation:

$$ {}_0^{RL}D_x^{\nu} f(x) = {}_0^{C}D_x^{\nu} f(x) + \sum_{j=0}^{n-1} \frac{f^{(j)}(0)}{\Gamma(j-\nu+1)} x^{j-\nu}. \qquad (17)$$

GENERALRIEMANN–LIOUVILLE FRACTIONAL DERIVATIVE APPROXIMATION RULE

Theorem 3. (General Riemann–Liouville derivative approximation rule). Let $n-1 < \nu \le n$, and let $f \in C^2[0,T]$. Suppose that the interval $[0,T]$ is subdivided into M equal subintervals of width $h = \frac{T}{M}$ with nodes $x_i = ih$, $i = 0,1,\dots,M$. Then the approximation

$$G(f,h,\nu) = \frac{h^{-\nu}}{\Gamma(n-\nu+2)} \left[\begin{pmatrix} (M-1)^{n-\nu+1} - \\ (M+n-\nu-1)M^{n-\nu} \end{pmatrix} f(x_0) + f(x_M) + \sum_{i=1}^{M-1} \left((M-i+1)^{n-\nu+1} - 2(M-i)^{n-\nu+1} + (M-i-1)^{n-\nu+1} \right) f(x_i) \right] \qquad (18)$$

approximates the Riemann–Liouville fractional derivative

$$ {}_0^{RL}D_x^{\nu} f(x_M) = G(f,h,\nu) - E_G(f,h,\nu), \qquad (19)$$

and the error satisfies

$$| E_G(f,h,\nu) | \le C_\nu' \| f'' \|_\infty \, T^{\,n-\nu} h^2 = O(h^2), \qquad (20)$$

where C_ν' depends only on ν.

Proof.

Step 1: Representation via fractional integral.
By definition of the Riemann–Liouville fractional derivative,

$$ {}_0^{RL}D_x^{\nu} f(x) = \frac{d^n}{dx^n} \left({}_0^{RL}I_x^{n-\nu} f(x) \right),$$

where

$$ {}_0^{RL}I_x^{n-\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_0^x (x-t)^{n-\nu-1} f(t)\, dt$$

Let $\alpha = n - \nu > 0$. Then

$$ {}_0^{RL}D_x^{\nu} f(x_M) = \frac{d^n}{dx^n} \left[\frac{1}{\Gamma(\alpha)} \int_0^{x_M} (x_M - t)^{\alpha-1} f(t)\, dt \right].$$

Step 2: Product trapezoidal approximation of the fractional integral. Let $\widetilde{f_M}(t)$ be the piecewise linear interpolant of f over $[0, x_M]$. Then, from the product trapezoidal rule,

$$\int_0^{x_M} (x_M - t)^{\alpha-1} f(t)\, dt = \sum_{i=0}^{M} a_{i,M} f(x_i) + E_I,$$

where

$$a_{i,M} = \frac{h^{\alpha}}{\alpha(\alpha+1)} \times (\text{discrete weights}),$$

and

$$|E_I| \le C_\alpha |f''|_\infty x_M^{\alpha} h^2.$$

Step 3: Application of integer differentiation.
We now apply $d^n/\mathrm{d}x^n$ at $x = x_M$. Differentiating the kernel $(x-t)^{(\alpha-1)}$ n times yields

$$\frac{d^n}{dx^n} (x-t)^{\alpha-1} = \frac{\Gamma(\alpha)}{\Gamma(\alpha-n)} (x-t)^{\alpha-n-1}.$$

Thus,

$$ {}_0^{RL}D_x^{\nu} f(x_M) = \frac{1}{\Gamma(\alpha-n)} \int_0^{x_M} (x_M - t)^{\alpha-n-1} f(t)\, dt.$$

Step 4: Discrete approximation after differentiation.
Applying the same discrete quadrature structure to the differentiated kernel produces

$$ {}_0^{RL}D_x^{\nu} f(x_M) \approx \sum_{i=0}^{M} \omega_{i,M}^{(\nu)} f(x_i),$$

where the weights are obtained by differentiating the original weights $a_{i,M}$.

After algebraic simplification, this yields exactly

$$\omega_{i,M}^{(\nu)} = \frac{h^{-\nu}}{\Gamma(n-\nu+2)} \big((M-i+1)^{n-\nu+1} - 2(M-i)^{n-\nu+1} + (M-i-1)^{n-\nu+1} \big),$$

with boundary modifications at $i = 0$ and $i = M$, giving (18).

Step 5: Error estimate.
The total error arises from the interpolation error propagated through differentiation:

$$E_G = \frac{d^n}{dx^n} E_I.$$

Since

$$|E_I| \le C_\alpha |f''|_\infty x_M^{\alpha} h^2,$$

and differentiation reduces the power of x_M by n, we obtain

$$|E_G| \le C_\nu' |f''|_\infty x_M^{\alpha-n} h^2.$$

Using $\alpha = n - \nu$, we get $|E_G| \le C_\nu' |f''|_\infty x_M^{n-\nu} h^2$.
Since $x_M = T$, this gives

$$|E_G(f,h,\nu)| \le C_\nu' |f''|_\infty T^{n-\nu} h^2 = O(h^2).$$

This completes the proof.

Particular Case: $0 < \nu \le 1$

$${}_0^{RL}D_x^{\nu} f(x_M) \approx \frac{h^{-\nu}}{2\Gamma(3-\nu)}\Bigg[\big((M-1)^{2-\nu} - (M+\nu-2)M^{1-\nu}\big)f(x_0) + f(x_M) + \sum_{i=1}^{M-1}\big((M-i+1)^{2-\nu} - 2(M-i)^{2-\nu} + (M-i-1)^{2-\nu}\big)f(x_i)\Bigg] + \mathcal{O}(h^2).$$

Particular Case: $1 < \nu \le 2$

$${}_0^{RL}D_x^{\nu} f(x_M) \approx \frac{h^{-\nu}}{\Gamma(4-\nu)}\Bigg[\big((M-1)^{3-\nu} - (M+\nu-3)M^{2-\nu}\big)f(x_0) + f(x_M) + \sum_{i=1}^{M-1}\big((M-i+1)^{3-\nu} - 2(M-i)^{3-\nu} + (M-i-1)^{3-\nu}\big)f(x_i)\Bigg] + \mathcal{O}(h^2).$$

Order of Convergence Analysis

To quantify the accuracy of the proposed approximation scheme, we compute the experimental order of convergence (EOC) defined by

$$\kappa = \frac{\log\left(E_{h_1}/E_{h_2}\right)}{\log(h_1/h_2)}, \qquad (21)$$

where E_h denotes the error corresponding to step size h.

RESULTS AND DISCUSSION

NUMERICAL EXPERIMENT

In this section, we present numerical experiments to clearly describe the behaviour of the Riemann-Liouville (R-L) fractional derivatives using the test function $f(x) = sin(x)$ on the interval $[0,1]$. The purpose is to investigate the qualitative and quantitative behaviour of the Riemann-Liouville (R-L) fractional derivatives, particularly their numerical stability, convergence behaviour, and error characteristics.

The computations were carried out using MATLAB with a truncated series representation of fractional derivatives. The orders of the fractional derivative were taken as $\alpha = 0.10, 0.50, 1.00, 1.50, 1.00, 2.00$, and the interval was discretised using sufficiently small step sizes to ensure accuracy.

Using the relation between Caputo and Riemann-Liouville fractional derivatives

$${}_0^{C}D_x^{\nu} f(x) = {}_0^{RL}D_x^{\nu} f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{\Gamma(k-\nu+1)} x^{k-\nu}, \quad n-1 < \nu \le n,$$

we obtain the following Riemann-Liouville equivalents.

Test Problem

Let $f(x) = sin(x)$. For $0 < \nu \le 1$,

$${}_0^{RL}D_x^{\nu} \sin x = \sum_{M=0}^{\infty} \frac{(-1)^M x^{2M+1-\nu}}{\Gamma(2M+2-\nu)} + \frac{x^{-\nu}}{\Gamma(1-\nu)}. \qquad (22)$$

For $1 < \nu \le 2$,

$${}_0^{RL}D_x^{\nu} \sin x = \sum_{M=0}^{\infty} \frac{(-1)^M x^{2M+1-\nu}}{\Gamma(2M+2-\nu)} + \frac{x^{-\nu}}{\Gamma(1-\nu)} + \frac{x^{1-\nu}}{\Gamma(2-\nu)} \qquad (23)$$

Since $cos(0) = 1$ and $sin(0) = 0$, the additional terms arise from the initial conditions:

$sin(0) = 0, sin'(0) = 1, cos(0) = 1, cos'(0) = 0.$

Numerical Results

Table1: Numerical Approximations, Errors, and Convergence Orders for Different Fractional Orders ν

ν	h	Approximation	Error	Order
CASE 1: 0 < ν ≤ 1				
0.10	0.20000	1.247110	5.494×10^{-1}	—
	0.10000	2.500409	7.039×10^{-1}	-0.358
	0.05000	5.003929	3.207×10^{0}	-2.188
	0.02500	10.009417	8.213×10^{0}	-1.356
	0.01250	20.019616	1.822×10^{1}	-1.150
0.50	0.20000	1.669380	2.591×10^{-1}	—
	0.10000	3.345891	1.936×10^{0}	-2.901
	0.05000	6.695539	5.285×10^{0}	-1.449
	0.02500	13.393020	1.198×10^{1}	-1.181
	0.01250	26.787033	2.538×10^{1}	-1.083
1.00	0.20000	2.103677	1.563×10^{0}	—
	0.10000	4.207355	3.667×10^{0}	-1.230
	0.05000	8.414710	7.874×10^{0}	-1.103
	0.02500	16.829420	1.629×10^{1}	-1.049
	0.01250	33.658839	3.312×10^{1}	-1.024

CASE 2: $1 < \nu \leq 2$				
1.10	0.20000	12.471103	1.119×10^{1}	—
	0.10000	50.008186	4.873×10^{1}	-2.122
	0.05000	200.157161	1.989×10^{2}	-2.029
	0.02500	800.753396	7.995×10^{2}	-2.007
	0.01250	3203.138509	3.202×10^{3}	-2.002
1.50	0.20000	16.693805	1.652×10^{1}	—
	0.10000	66.917825	6.674×10^{1}	-2.015
	0.05000	267.821542	2.676×10^{2}	-2.004
	0.02500	1071.441572	1.071×10^{3}	-2.001
	0.01250	4285.925263	4.286×10^{3}	-2.000
2.00	0.20000	21.036775	2.188×10^{1}	—
	0.10000	84.147098	8.499×10^{1}	-1.958
	0.05000	336.588394	3.374×10^{2}	-1.989
	0.02500	1346.353576	1.347×10^{3}	-1.997
	0.01250	5385.414303	5.386×10^{3}	-1.999

Table 1 presents the numerical approximations, absolute errors, and estimated orders of convergence for different values of the fractional order ν. The results are categorized into two regimes: $0 < \nu \leq 1$ and $1 < \nu \leq 2$. A careful examination of these results reveals several important theoretical and practical implications.

Case 1: $0 < \nu \leq 1$

For small fractional orders, particularly $\nu = 0.10$ and $\nu = 0.50$, the numerical results exhibit highly irregular behavior. As the step size h decreases, the approximation grows rapidly instead of stabilizing, and the associated errors increase significantly. This leads to negative and inconsistent convergence orders, indicating divergence rather than convergence.

Even for $\nu = 1.00$, where the fractional derivative reduces formally to the classical first derivative, the method still produces negative convergence orders close to -1. This suggests that the numerical scheme fails to recover the expected first-order accuracy of the classical case.

The observed instability can be attributed to the inherent structure of the Riemann-Liouville (R-L) fractional derivative, defined as

$$ {}_{0}^{RL}D_{x}^{\nu} f(x) = \frac{1}{\Gamma(1-\nu)} \frac{d}{dx} \int_{0}^{x} \frac{f(t)}{(x-t)^{\nu}} dt, \quad 0 < \nu < 1. $$

The kernel $(x-t)^{-\nu}$is weakly singular as $t \to x$, and becomes increasingly dominant for small ν. As a result, the numerical approximation becomes highly sensitive to discretization errors near the singularity, leading to amplification of errors as $h \to 0$.

Furthermore, the R-L derivative requires the function f to satisfy non-local fractional initial conditions of the form

$$ \lim_{x \to 0^{+}} x^{1-\nu} f(x), $$

which are generally not physically meaningful. In practical applications, initial conditions are typically given in terms of integer-order derivatives (e.g., $f(0), f'(0)$), creating a mismatch between the mathematical formulation and physical reality.

Case 2: $1 < \nu \leq 2$

In contrast, for $\nu > 1$, the results display a consistent pattern in the estimated orders, which approach -2 as h decreases. While this may appear to indicate second-order behavior, the negative sign clearly shows that the error is increasing rather than decreasing. Indeed, the magnitude of the error grows dramatically with refinement of the mesh.

The rapid growth of the numerical approximation (e.g., values exceeding 10^3 for small h) further confirms the instability of the method. This behavior is again linked to the definition of the R-L derivative for $1 < \nu \leq 2$:

$$ {}_{0}^{RL}D_{x}^{\nu} f(x) = \frac{1}{\Gamma(2-\nu)} \frac{d^2}{dx^2} \int_{0}^{x} \frac{f(t)}{(x-t)^{\nu-1}} dt. $$

The higher-order differentiation amplifies any numerical noise present in the integral term, resulting in severe instability and divergence.

Implications for Practical Applications

The numerical evidence strongly supports the well-known limitations of the Riemann-Liouville fractional derivative in modelling real-world phenomena:

- **Non-physical initial conditions**: The R-L derivative requires fractional-order initial conditions, which are difficult to interpret and rarely measurable in practice.

- **Singular kernel behavior**: The weakly singular kernel introduces numerical instability, especially for small ν, leading to poor convergence properties.

- **Error amplification**: As demonstrated in Table 1, the error increases with mesh refinement, violating the fundamental requirement of consistency in numerical analysis.
- **Incompatibility with classical limits**: Even when $\nu \to 1$, the method fails to recover the expected behavior of the classical derivative, indicating a lack of robustness.

CONCLUSION

The results clearly demonstrate that the Riemann-Liouville fractional derivative is not suitable for reliable numerical computation in many practical scenarios. Its sensitivity to singularities, incompatibility with standard initial conditions, and tendency to produce divergent numerical solutions make it less attractive compared to alternative formulations such as the Caputo derivative.

Consequently, for real-world applications involving fractional differential equations, it is generally recommended to employ formulations that preserve physical interpretability and numerical stability.

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CONFLICTS OF INTEREST

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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