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Two Adaptive RMIL-TYPE Conjugate Gradient Parameters Derived by Minimizing ℓ₁ and ℓ∞ Norm Condition Numbers of its Search Direction Matrix

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ABSTRACT
This study introduces two adaptive scaling parameters for the Rivaie–Mustafa–Ismail–Leong (RMIL) type conjugate gradient framework designed for unconstrained optimization problems. The proposed parameters are obtained by minimizing the condition numbers of the RMIL-type search direction matrix under the $l_1$and l_∞matrix norms, respectively. This minimization strategy is intended to enhance the conditioning and numerical stability of the search direction, thereby improving the overall performance of the conjugate gradient algorithm. The resulting adaptive parameters, denoted by t_k^1 and t_k^∞, are expressed in explicit form using gradient differences y_k and search directions d_k. To assess performance, numerical experiments are conducted on 57 CUTEr benchmark problems with dimensions varying from 50 to 100,000. The results are evaluated using performance profiles based on iteration counts, total function and gradient evaluations, and CPU time. The computational outcomes indicate that the t_k^∞parameter leads to a marked improvement in the efficiency of the RMIL-type method, whereas the t_k^1 parameter yields comparatively weaker performance. Overall, the findings highlight the effectiveness of adaptive parameter selection driven by matrix norm conditioning in conjugate gradient methods.

Keywords:
Conjugate gradient Method;
RMIL method;
Condition Number;
ℓ_1 norm; ℓ_∞ norm;
Unconstrained Optimization;
CUTEr benchmark.

INTRODUCTION

We study the unconstrained optimization problem

$$\min_{x\in\mathbb{R}^n} f(x), \tag{1}$$

where f is a smooth objective function. One widely used approach for solving problem (1) is the conjugate gradient (CG) scheme. The CG scheme is said to be attractive due to its ability for low memory requirements and favorable convergence behavior, which make it particularly effective for large-scale unconstrained optimization problems (Sun and Yuan, 2006; Dai et al., 1999).

The CG scheme generates a sequence of iterates in the form

$$x_0 \in \mathbb{R}^n, x_{k+1} = x_k + s_k, \text{with } s_k = \alpha_k d_k, k = 0,1,2,3,\dots \tag{2}$$

Here, $x_k$and x_{k+1} represents the previous and current iterates, respectively. The scalar $\alpha_k > 0$represents the step length, which may be obtained through either exact or inexact line search procedures.

The CG method generates search directions iteratively in the form

$$d_0 = -g_0, k = 0, \quad d_{k+1} = -g_{k+1} + \beta_k d_k, k = 1,2,3,\dots \tag{3}$$

Here, β_k is called the CG parameter (Hager & Zhang, 2006a). This parameter is essential in influencing the efficiency and convergence of the method. Various established formulas for β_k, with $g_k = \nabla f(x_k)$, are available in the literature.

$$\beta_k^{\text{HS}} = \frac{g_{k+1}^T(g_{k+1}-g_k)}{d_k^T(g_{k+1}-g_k)}, \quad \text{(Hestenes \& Stiefel, 1952)}$$

$$\beta_k^{\text{PRP}} = \frac{g_{k+1}^T(g_{k+1}-g_k),}{\|g_k\|^2} \quad \text{(Polak \& Ribiere, 1969)}$$

$$\beta_k^{\text{LS}} = \frac{g_{k+1}^T(g_{k+1}-g_k)}{-d_k^T g_k}, \quad \text{(Liu \& Storey, 1991)} \tag{4}$$

For detailed reviews of conjugate gradient methods and their different extensions, the reader may consult,

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for instance, (Hager & Zhang, 2006a; Narushima & Yabe, 2014). In practice, inexact line search techniques are generally preferred over exact line search due to their computational efficiency. Among these, the weak Wolfe and strong Wolfe conditions are the most commonly applied inexact line search strategies. The weak Wolfe line search determines the step size $\alpha_k$such that it satisfies the following conditions:

$$f(x_k + \alpha_k d_k) \leq f(x_k) + \delta \alpha_k g_k^T d_k, \\ g(x_k + \alpha_k d_k)^T d_k \geq \sigma g_k^T d_k \qquad (5)$$

and the strong Wolfe line search is calculated such that α_k satisfy:

$$f(x_k + \alpha_k d_k) \leq f(x_k) + \delta \alpha_k g_k^T d_k, \quad |g(x_k + \alpha_k d_k)^T d_k| \ \leq -\sigma g_k^T d_k \qquad (6)$$

where $g_k = \nabla f(x_k)$ is a gradient f at point x_k , and the parameters δ and σ are required to satisfy
$0 < \delta < \sigma < 1$.

One fundamental requirement for any CG approach is the global convergence and this property is closely related to the descent condition

$$d_k^T g_k < 0 \qquad (7)$$

or the sufficient descent condition which is of the form

$$d_k^T g_k \leq -c \parallel g_k \parallel^2, c > 0, \qquad (8)$$

where ‖·‖denotes the Euclidean norm.

Rivaie *et al.* (2012) introduced the Rivaie–Mustafa–Ismail–Leong (RMIL) conjugate gradient scheme for addressing (1), in which the denominator in formula (4) was modified by substituting it with $\parallel d_k \parallel^2$.

Within this framework, the CG parameter is defined in the form

$$\beta_k^{\text{RMIL}} = \frac{g_{k+1}^T (g_{k+1} - g_k)}{\parallel d_k \parallel^2}. \qquad (9)$$

By employing exact line search, the authors demonstrated that RMIL approach fulfills the sufficient descent condition. However, this property does not generally hold when inexact line search techniques are used, and global convergence is guaranteed only in exact line search setting. This restriction reduces its effectiveness in practical applications (Dai, 2016). As a result, more recent research has concentrated on improving the RMIL framework by incorporating adaptive mechanisms or inexact line search strategies to enhance its stability and convergence performance (Salihu et al., 2023; Yousif, 2020; Sun, 2006).

Moyi et al. (2024) developed an improved nonlinear conjugate gradient approach for addressing large-scale unconstrained optimization problems. Their work enhanced the classical conjugate gradient framework by introducing a parameter that ensures the sufficient descent condition without relying on the line search strategy. In addition, they proved global convergence of the method under strong Wolfe line search assumptions. The effectiveness of the proposed algorithm was further verified through numerical experiments on standard benchmark problems. The results indicated that the modified method performs competitively with existing conjugate gradient techniques in terms of iteration numbers, CPU time, and overall robustness.

Recently, Ahmed et al. (2024) suggest a new RMIL type approach for solving monotone systems of nonlinear equations with search direction of the form

$$d_0 = -g_0, \ k = 0, \quad d_{k+1} = -F_{k+1} + \frac{F_{k+1}^T y_k}{\|d_k\|^2} d_k - t_k \frac{F_{k+1}^T d_k}{\|d_k\|^2} d_k, \qquad (10)$$

$k \geq 1$,.

From a matrix-based perspective, Ahmed et al. (2024) introduced several adaptive strategies for selecting the parameter t_k. In particular, by adopting the approach of Perry (1976), they demonstrated that the search direction matrix related to the two RMIL-type methods can be presented in the form

$$d_{k+1} = -A_{k+1} g_{k+1}$$

in which

$$A_{k+1} = I - \frac{d_k y_k^T}{\|d_k\|^2} + t_k \frac{d_k d_k^T}{\|d_k\|^2} \qquad (11)$$

The symmetrized version of $A_{k+1}$is given by

$$R_{k+1} = I - \frac{1}{2} \frac{d_k y_k^T + y_k d_k^T}{\|d_k\|^2} + t_k \frac{d_k d_k^T}{\|d_k\|^2}. \qquad (12)$$

The work of Ahmed *et al.*, (2024) further showed that the descent property is guaranteed when

$$t_k = m \frac{\|y_k\|^2}{\|d_k\|^2} + n \left(\frac{1}{2} \frac{d_k^T \bar{y}_k}{\|d_k\|^2} - 1 \right)^2, \qquad (13)$$

where $m > \frac{1}{4}$ and $n \geq 0$, which ensures that $R_{k+1}$is positive definite.
Here, we develop two adaptive choices for t_k in (10) under unconstrained optimization using the concept of l_1 and l_∞ norms condition number minimization. Section 2 focused on deriving the two adaptive choices from the

RMIL-type search direction matrix A_{k+1} as well as the convergence result of the new scheme. Section 3 demonstrate numerical experiments of the new scheme on Constrained and Unconstrained Testing Environment, revisited (CUTEr) benchmark problems. Section 4 concludes the study and outlines potential avenues for further investigation.

MATERIALS AND METHODS

Two Adaptive Choices of Rmil-type Parameter Based on l_1 and l_∞ Norms Condition Number Minimization

In this section, we develop two adaptive formulations of the Rivaie–Mustafa–Ismail–Leong (RMIL)-type parameter by minimizing the condition number with respect to the $\ell_1$and ℓ_∞norms. Drawing inspiration from quasi-Newton methods, Ahmed *et al.* (2024), based on the Dai and Liao approach, suggest a modified RMIL-type scheme for solving monotone systems of nonlinear equations of the form $F(x) = 0$. Since unconstrained optimization problems can be equivalently expressed as nonlinear systems by letting $F(x) = \nabla f(x)$, this framework can be directly extended to unconstrained optimization problems. With this formulation, the RMIL-type conjugate gradient parameter is given by:

$$\beta_k = \frac{g_{k+1}^T y_k}{\|d_k\|^2} - t_k \frac{g_{k+1}^T d_k}{\|d_k\|^2}, k \geq 1, \tag{14}$$

with search direction

$$d_0 = -g_0,\ k = 0,\quad d_{k+1} = -g_{k+1} + \frac{g_{k+1}^T y_k}{\|d_k\|^2} d_k - t_k \frac{g_{k+1}^T d_k}{\|d_k\|^2} d_k \tag{15}$$

where $y_k = g_{k+1} - g_k$ *and* $t_k \geq 0$ is an adaptive parameter extending the classical RMIL scheme.

To formulate a new adaptive scaling approach, certain fundamental properties of the matrix l_1-norm and l_∞-norm are first considered. Based on these norm characteristics, an appropriate RMIL-type parameter is constructed through the minimization of the l_1-norm condition number associated with the search direction matrix A_{k+1}given in (11). Likewise, an alternative optimal selection of the parameter $t_k$is obtained by minimizing the l_∞-norm condition number of the same matrix A_{k+1}.

It is well known that for any matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$,

$$\| A \|_1 = \max_{j=1,2,\dots,n} \sum_{i=1}^{n} | a_{ij} |, \| A \|_\infty = \max_{i=1,2,\dots,n} \sum_{j=1}^{n} | a_{ij} |.$$

Moreover, if $A = xy^T$, where $x, y \in \mathbb{R}^n$are arbitrary vectors, then

$$\| A \|_1 = \| x \|_1 \| y \|_1, \| A \|_\infty = \| x \|_\infty \| y \|_\infty.$$

For a nonsingular matrix A, the ℓ_1 and ℓ_∞-norm condition numbers are defined, respectively, as

$$\kappa_1(A) = \| A \|_1 \| A^{-1} \|_1, \kappa_\infty(A) = \| A \|_\infty \| A^{-1} \|_\infty.$$

These condition numbers are important measures of the sensitivity of numerical solutions to perturbations in data. In particular, a matrix with a large condition number is said to be *ill-conditioned*, indicating that numerical instability may arise in computations involving such a matrix.

Motivated by this idea, the parameter $t_k$in (11) is chosen by minimizing an upper bound that simultaneously controls both the $l_1$and l_∞norm condition numbers of the matrix A_{k+1}. This approach guarantees that the resulting matrix maintains good conditioning throughout the iterative process.

Based on the preceding discussion, the ℓ_1-norm of A_{k+1}can be bounded as

$$\| A_{k+1} \|_1 \leq 1 + \frac{\| d_k \|_1 \| y_k \|_\infty}{\|d_k\|^2} + t_k \frac{\| d_k \|_1 \| d_k \|_\infty}{\|d_k\|^2}.$$

Furthermore, by applying the Sherman–Morrison formula, we obtain

$$A_{k+1}^{-1} = I + \frac{d_k y_k^T}{\|d_k\|^2(1 + t_k) - d_k^T y_k} - t_k \frac{d_k d_k^T}{\|d_k\|^2(1 + t_k) - d_k^T y_k}.$$

Consequently, its ℓ_1-norm satisfies the bound

$$\| A_{k+1}^{-1} \|_1 \leq 1 + \frac{\| d_k \|_1 \| y_k \|_\infty}{\|d_k\|^2(1 + t_k) - d_k^T y_k} + t \frac{\| d_k \|_1 \| d_k \|_\infty}{\|d_k\|^2(1 + t_k) - d_k^T y_k}.$$

Combining these estimates, we obtain the following upper bound for the condition number:

$$\kappa_1(A_{k+1}) \leq h(t) + \zeta,$$

where $h(t)$is a function depending on t, and $\zeta$is a constant independent of t.

where $\zeta$is a positive constant, and the function $h(t)$is defined by

$$h(t) = t_k \| d_k \|_\infty \left(\frac{\| d_k \|_1}{\|d_k\|^2} + \frac{\| d_k \|_1}{\|d_k\|^2(1 + t_k) - d_k^T y_k} \right) + \frac{2 \| d_k \|_1}{\|d_k\|^2} \left(\frac{\| d_k \|_1}{\|d_k\|^2(1 + t_k) - d_k^T y_k} \right) (\| d_k \|_\infty)^2 {t_k}^2, t_k > 0.$$

After some algebraic simplifications, the optimal parameter

$$t_k^1 = \arg \min_{t>0} h(t)$$

is obtained as

$${t_k}^1 = \sqrt{\left(\frac{\|y_k\|_\infty}{\|d_k\|_\infty} \right) \left(\frac{\|d_k\|^2 + \|d_k\|_1 \|y_k\|_\infty}{\|d_k\|^2 + \|d_k\|_1 \|d_k\|_\infty} \right)}$$

which minimizes the upper bound of the ℓ_1-norm condition number given in (2.1).

Similarly, by carrying out an analogous analysis using the ℓ_∞-norm, we obtain

$$t_k^{\infty} = \sqrt{\left(\frac{\|y_k\|_1}{\|d_k\|_1}\right)\left(\frac{\|d_k\|^2 + \|d_k\|_\infty \|y_k\|_1}{\|d_k\|^2 + \|d_k\|_\infty \|d_k\|_1}\right)} \quad (17)$$

as a minimizer of an upper bound of $\kappa_\infty(A_{k+1})$.

Under the standard assumptions of bounded level sets, Lipschitz continuous gradients, and the strong Wolfe line search conditions, the proposed adaptive scaling parameters t_k^1 and t_k^∞ preserve the descent property of the RMIL-type conjugate gradient method. Since both parameters satisfy the positive definiteness condition established by Ahmed et al., (2024) (with $\bar{y} = y_k$), the generated search directions remain descent directions for all iterations. Consequently, by the Zoutendijk theorem, the proposed methods satisfy the global convergence condition

$$\liminf_{k\to\infty} \| g_k \| = 0.$$

Algorithm

STEP 1: Given $x_0 \in \mathbb{R}^n,\ \epsilon = 10^{-6},\ maxit = 2000$ and set $k = 0$.
STEP 2: Compute $\nabla f(x_k)$
STEP 3: Check if $\|\nabla f(x_k)\| \le \epsilon$ or maxit ≥ 2000, stop else go to step 4.
STEP 4: Compute α_k via the line search in (Strong Wolf)
STEP 5: Compute t_k via the scaling parameter in (16) or (17)
STEP 4: Compute the direction d_k by (15)
STEP 6: Set $x_{k+1} = x_k + s_k$, where $s_k = \alpha_k d_k$.
STEP 7: Set $k = k + 1$ and go to step 2.

RESULTS AND DISCUSSION

Numerical Results

This section presents the numerical results carried out to support the theoretical findings. The evaluation begins with a series of tests on both constrained and unconstrained problems drawn from the Constrained and Unconstrained Testing Environment, revisited (CUTEr) benchmark set in table 1. In table 1, the experiments involve 57 test problems, each having dimension $n \ge 50$. All computations were implemented in MATLAB version 7.14.0.739 (R2012a), running on a T520 Core i7 Lenovo desktop (DESKTOP-VGA6K96) with a 64-bit operating system, 2.80 GHz processor, and 8 GB of RAM. Further details of the test problems are summarized in same Table 1 below:

TABLE 1: TEST PROBLEM DATA

FUNCTION	*DIM(N)*	*FUNCTION*	*DIM(N)*	*FUNCTION*	*DIM(N)*
EPenlty	*100*	*raydan1*	*100*	*DENSCHNA*	*1000*
EPenlty	*1000*	*raydan1*	*50*	*DENSCHNA*	*10000*
EPenlty	*2000*	*raydan2*	*10000*	*HIMMELBH*	*10000*
Diag5	*50000*	*raydan2*	*50000*	*HIMMELBH*	*10000*
Diag5	*100000*	*raydan2*	*100000*	*HIMMELBH*	*10000*
QP1	*100*	*FLETCHCR*	*1000*	*DQDRTIC*	*1000*
QP1	*10000*	*FLETCHCR*	*50000*	*DQDRTIC*	*10000*
QP2	*500*	*eDENSCHNB*	*100*	*DQDRTIC*	*100*
QF2	*50*	*eDENSCHNB*	*5000*	*QUARTICM*	*1000*
QF2	*1000*	*eDENSCHNB*	*10000*	*QUARTICM*	*10000*
QF2	*5000*	*Diag6*	*10000*	*LPerturbed*	*5000*
Diag2	*100000*	*Diag6*	*5000*	*LPerturbed*	*10000*
SumSqr	*100*	*Diag6*	*10000*	*LPerturbed*	*20000*
SumSqr	*2000*	*Diag6*	*50000*	*DENSCHNF*	*1000*
SumSqr	*5000*	*Diag4*	*1000*	*DENSCHNF*	*10000*
Shallow	*1000*	*Diag4*	*10000*	*DENSCHNF*	*50000*
Shallow	*10000*	*Diag4*	*100000*	*ARWHEAD*	*500*
QUARTC	*100*	*Diag7*	*100*	*exhimmelblau*	*500*
QUARTC	*1000*	*Diag7*	*100*	*exhimmelblau*	*1000*

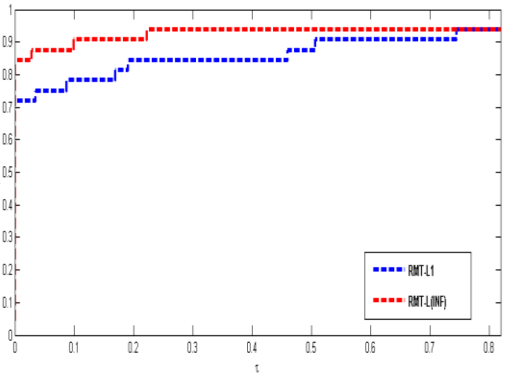
Aminifard & Babaie-kafaki ***(2019): Constrained and Unconstrained Testing Environment, revisited (CUTEr)***

To assess the performance of the parameters $t_k^1$and t_k^{∞}within the Rivaie–Mustafa–Ismail–Leong (RMIL)-type framework where the search direction is defined by (3) a comparative analysis of the corresponding methods was conducted. These approaches utilize the scaling parameters specified in (16) and (17), respectively.

For the proposed algorithm, the step size was determined using a Wolfe-type line search with parameters $c_1 = 10^{-4}$and $c_2 = 0.5$. The iterative process was terminated once a solution satisfying $\| \nabla f(x_k) \| \leq 10^{-6}$was reached, or when the iteration limit of $k = 2000$was attained.

To evaluate the performance of the algorithms, the Dolan–Moré performance profile (2007) was employed. The comparison was based on NI, TNFGE, and CPU time, which respectively denote the Number of Iterations and the Total Number of Function and Gradient Evaluations, as described in Hager and Zhang (2006b), along with the computational time. The comparative performance results are presented in Figures 1–3.

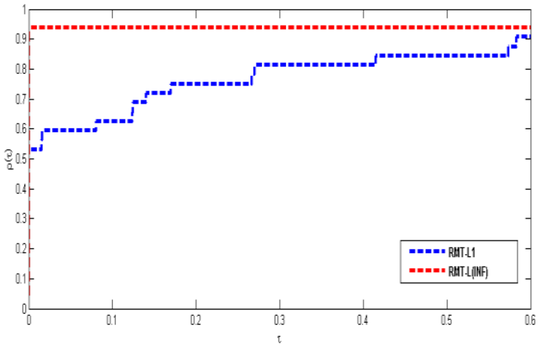
The findings show that RMT-L(INF), which is derived from the ℓ_∞-norm (i.e., t_k^{∞}), significantly improves the efficiency of the RMIL-type method. On the other hand, RMT-L1, based on the ℓ_1-norm (i.e., t_k^1), does not yield an effective scaling strategy for the RMIL-type approach.



Dolan & Moré, (2007)

Figure 1: Performance profile for number of iteration (NI)

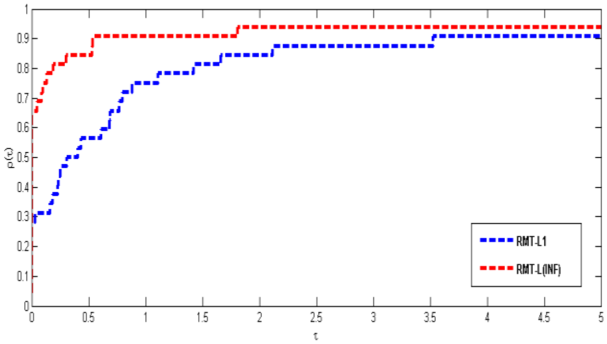
Figure 1 illustrates the Dolan–Moré performance profile comparing the proposed adaptive scaling strategies $t_k^{(1)}$RMT-L1 and t_k^{∞}RMT-L(INF) using the Number of Iterations (NI) as the performance metric. In the profile, the blue line denotes $t_k^{(1)}$, whereas the red line represents t_k^{∞}. The graph shows that t_k^{∞}attains a better performance across a wider range of the tested problems, indicating greater computational effectiveness. This behavior implies that the adaptive parameter constructed from the l_∞-norm achieves quicker convergence and yields more efficient numerical results than the l_1-norm based parameter $t_k^{(1)}$.



Dolan & Moré, (2007)

Figure 2: Performance profile for total number of function and gradient evaluation (TNFGE)

Figure 2 displays the Dolan–Moré performance profile for the adaptive scaling parameters $t_k^{(1)}$RMT-L1 and t_k^{∞}RMT-L(INF) with respect to the Total Number of Function and Gradient Evaluations (TNFGE). The blue profile corresponds to $t_k^{(1)}$, whereas the red profile denotes t_k^{∞}. From the plot, it can be seen that t_k^{∞}performs more efficiently on a greater percentage of the benchmark problems, as reflected by its dominant performance curve. This indicates that the l_∞-norm based adaptive scaling parameter attains improved convergence characteristics and better overall numerical efficiency when compared with the l_1-norm based parameter $t_k^{(1)}$.



Dolan & Moré, (2007)

Figure 3: Performance profile for CPU time (CPUT)

Figure 3 shows the Dolan–Moré performance profile of the proposed adaptive scaling parameters $t_k^{(1)}$RMT-L1 and t_k^{∞}RMT-L(INF) using CPU time (CPUT) as the evaluation criterion. The blue curve represents $t_k^{(1)}$, while the red curve corresponds to t_k^{∞}. The figure reveals that t_k^{∞}achieves better computational performance across a larger set of the test problems, as evidenced by its more dominant profile. This outcome suggests that the adaptive parameter based on the l_∞-norm exhibits faster convergence and enhanced numerical efficiency relative to the l_1-norm based parameter $t_k^{(1)}$.

CONCLUSION

This paper developed two adaptive scaling parameters for the Rivaie–Mustafa–Ismail–Leong (RMIL) type conjugate gradient method by minimizing the l_1 and l_∞ norm condition numbers of the search direction matrix. Numerical experiments on CUTEr benchmark problems demonstrated that the RMT-L(INF) parameter (based on the l_∞-norm) effectively enhances the performance of the RMIL-type method, while the RMT-L1 parameter proved less satisfactory. These findings provide a norm-minimization perspective for improving Conjugate Gradient method robustness. Future work may explore extensions to nonconvex optimization and other large-scale applications.

Conflict of Interest: No conflict of interest to declare

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