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A Self-Starting Block Methods (SSBMs) for the Solution of Ordinary Differential Equations

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ABSTRACT
In this paper, self-starting block numerical methods for the solution of stiff initial value problems ordinary differential equations were developed. The Backward Differentiation Formulas and Generalized Backward Differentiation Formulas are used in the derivations. The E-transformation is applied to the triples and self-starting methods are obtained. The numerical implementation of the methods on ordinary differential equations are reported to show the effectiveness and efficiency of the methods. The computational of convergence of absolute stability and consistent tends to the theoretical order as h tends to zero.

Keywords:
Block methods,
Multistep methods,
Ordinary differential Equations

INTRODUCTION

In this paper, the derivation of a self-starting block methods for the numerical solution *y*(*t*) of the initial value problem is shown

$$y'(t) = f(t, y(t)); y(t_0) = y_0;\ t \in [a,b]; \quad f : \Re \times \Re^m \to \Re^m; y : \Re \to \Re^m \tag{1}$$

The Reversed Adams Moulton (RAM) methods are employed and generally written as

$$y_1 - y_0 = h_n \sum_{i=0}^{k} \beta_r f_{n+r} \tag{2}$$

(see Brugnano and Trigiante, 1998).
The matrix equations of the form:

$$\begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{3} \\ . \\ . \\ . \\ \frac{1}{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1. . .1 \\ 0 & 1 & 2 & 3. . .k \\ 0 & 1 & 2^2 & 3^2. . .k^2 \\ . & . & . & . . . . . \\ . & . & . & . . . . . \\ . & . & . & . . . . . \\ 0 & 1 & 2^k & 3^k. . .k^k \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ . \\ . \\ . \\ \beta_k \end{pmatrix} \tag{3}$$

was used to solve simultaneously for the coefficients. They are therefore generally zero stable. The determination of the coefficients $\{\beta\}_{r-0}^{k}$ is done by imposing the maximum order $k+1$ on the method (2) (see Ajie, Ikhile & Onumanyi (2013). Brugnano & Trigiante (1998) and several authors such as Li, Wen, & Zhang (2019); Noor, Ibrahim & Ismail (2018); Öztürk, (2018); Adesanya, Fotta & Abdulkadri (2015); Sagir (2014); Yahaya & Sagir (2013); Sagir (2012); Brugnano & Trigiante (2001) have written on self-starting block methods. In a paper of Abdullahi, Suleiman, Sagir & Sule, (2022) an A- stable block integrator scheme for the solution of first order system of IVP of ordinary differential equations was developed and have advantages in terms of accuracy, minimum errors and less computational time. Others include Sagir, Abdullahi, & Muhammad (2023); (Sagir & Abdullahi 2022; Sagir & Abdullahi 2023).

Definition 1: A block method is said to be pre-stable if the roots of $Q(z)$ are contained in C^+.
For the cases of orders 3, 5 and 7 above $D(z)$ has no negative pole on C^-. In all the cases, the roots of $Q(z)$ are contained in C^+

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Definition 2: A numerical IVP is A-stable if its region of absolute stability includes the entire complex half-plane with negative real part,C−. A method is L-stable if it is A-stable; hence L-stability is a special case of A-stability.

Definition 3: The block method is *A*-stable if and only if it is stable on the imaginary axis (*I*-stable): $D(iy) \leq 1$ for all $y \in \Re$, and *D*(z) is analytic for $D(z) < 0$ (i.e., $Q(z)$ does not have roots with negative or zero real parts), *I*-stability is equivalent to the fact that the Norsett polynomial defined by

$$E(y) = |Q(iy)|^2 - |P(iy)|^2 = Q(iy)Q(-iy) - P(iy)P(-iy) \quad (8)$$

satisfies $E(y) > 0$ for all $y \in \Re$, see (Jackiewicz, 2009). In each of the cases of order *p* = 3, 5, 7, (8) is satisfied and $D(z) \to 0$ as $z \to \infty$ implying that the methods are *L-stable* for $k \leq 7$.

MATERIALS AND METHODS

Derivation of the Method

The Backward Differentiation Formulas (BDF)

A *k*-step BDF introduced in Curtiss and Hirschfelder (1952) is a linear multistep formula that has order *p* = *k* and error constant $C_{p+1} = \dfrac{-1}{k+1}$ when the coefficient of the derivative function is normalized to one. They are popular for the solution of stiff differential equations (1). They have the general formula

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h_n \beta_k f_{n+k} \quad (4)$$

The coefficients $\{\alpha_j\}_{j=1}^{k}$ are uniquely determined by imposing the order *k* on (4) which leads to the matrix equation

$$\begin{pmatrix} 0 \\ 1 \\ 2k \\ . \\ . \\ . \\ k^k \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 . . . 1 \\ 0 & 1 & 2 & 3 . . . k \\ 0 & 1 & 2^2 & 3^2 . . . k^2 \\ . & & & \\ . & & & \\ . & & & \\ 0 & 1 & 2^k & 3^k . . . k^k \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ . \\ . \\ . \\ \alpha_k \end{pmatrix} \quad (5)$$

which are solved simultaneously. The methods have been shown in Ajie, Ikhile and Onumanyi (2013), Lambert (1991) and Brugnano and Trigiante (1998) to be zero stable for $k \leq 6$, and zero unstable for $k \geq 7$.

The Generalized Backward Differentiation Formulae (GBDF)

This class of methods introduced in Brugnano and Trigiante (1998) has the form

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h f_{n+j} \quad (6)$$

for all $k \geq 1$, where

$$j = \begin{cases} \dfrac{k+2}{2}, & for\ even\ k, \\ \dfrac{k+1}{2}, & for\ odd\ k \end{cases} \quad (7)$$

The Generalized Backward Differentiation Formulae (6) though cannot be used as single integrator, provide $0_{j,\,k-j}$ - Stable, $A_{j,\,k-j}$ - Stable methods for all $k \leq 32$.

The methodology for the derivation is captured in the following theorem Ajie, Utalor & Onumanyi (2019).

Theorem

Let the multi-family of Linear multistep formula (LMF) $\left\{\rho_k^{[j]}(R),\ \sigma_k^{[j]}(R)\right\}_{j=1,\,k=1}^{m,\,K}$ *be given, that is,*

$$\rho_k^{[j]}(E) y_n = h \sigma_k^{[j]}(E) f_n;\ j = 1(1)m,\ k = 1(1)K \quad (9)$$

with $\left\{\rho_k^{[j]}, \sigma_k^{[j]}\right\}$ for a fixed j forming a family of variable order $p_{k,\,j}$ of variable step number k. Then the resultant system of composite LMF

$$E^i \rho_k^{[j]}(E) y_n = h E^i \sigma_k^{[j]}(E) f_n; \\ i = 0(1)k - l;\ j = 1,\ 2, ..., m \quad (10)$$

Arising from the E-operator transformation of (8) can be composed as the block method

$$A_1 Y_{n+1} + A_0 Y_n = h(B_1 F_{n+1} + B_0 F_n);\ \det(A_1) \neq 0 \quad (11)$$

if k is chosen such that l is an integer given as

$$l = \frac{m + k(m-2)}{m-1}\ ;\ m, k \geq 2 \text{ and } k - l \geq 0. \quad (12)$$

where Y_{n+1}, Y_n; F_{n+1} *and* F_n

$n = 0,1,2,\ldots$ *are vectors and* A_1, A_0, B_1, B_0 are square matrices for a fixed m.

Proof:

The *E*-operator is effectively applied *k-l* times on the system of LMF $\left\{\rho_k^{[j]}, \sigma_k^{[j]}\right\}_{k,j}$. Thus, there are 2*k-l* unknown solution points captured in the block of solution $Y_{n+1} = (y_{n+1}, y_{n+2}, \ldots, y_{n+2k-l})^T$. By this the block definition in (12) is realized if the coefficient matrices A_1, A_0, B_1, B_0 are square matrices of dimension $(2k-l)\times(2k-l)$.

This simply imply that $m + m(k-l) = 2k - l$ so that *l* is as in (12) and for a fixed *m* the *k* is then chosen such that $k - l \geq 0$

In particular: $m = 3$; $l = \frac{k+3}{2}$; $k = 3,\ 5,\ 7, \ldots$

When $k - l = 0$, the method requires no shifting, this is so if $m = k$.

Consider the triple of *k*-step LMF defined by $[\rho_1, \sigma_1]$, $[\rho_2, \sigma_2]$ and $[\rho_3, \sigma_3]$, shifting this (*k-l*) times, where $l = \frac{k+3}{2}$, a set of 2*k-l* equations in 2*k-l* unknowns which can be written in the block form (10) have to be obtained.

Stability of the Implicit Block Methods

When (11) is applied to test equation

$$y' = \lambda y, \mathrm{Re}(\lambda) < 0 \tag{13}$$

it yields the characteristics equation

$$\pi(R,z) = \det(A_1 R + A_0 - z(B_1 R + B_0)) \tag{14}$$

The region of absolute stability R_A associated with (11) is the set

$$R_A = \{z : |R_j(z)| \leq 1,\ j = 1(1)k\} \tag{15}$$

If $z \to 0$ in (14), the difference system becomes

$$\pi(R,0) = \det(A_1 R + A_0) \tag{16}$$

All the proposed block methods can be presented in the form:

$$A_1 Y_{n+1} + \hat{a} y_n = h(B_1 F_{n+1} + \hat{b} f_n) \tag{17}$$

where $\hat{a}$ & $\hat{b}$ are vectors of $(2k-i)$ x 1 and $A_1^{-1}\hat{a} = (111\ldots1)^T = e$

To see this, assume order $p \geq 1$ for all the LMF that constitute the block, then by consistency,

$$A_1 e + \hat{a} = 0 \tag{18}$$

From (18) it follows that

$$A_1^{-1}\hat{a} = -e \tag{19}$$

The above ensures zero-stability of the implicit block methods (11), which if applied to test equation can also be written as:

$$Y_{n+1} = M(z)Y_n,\ z = \lambda h \tag{20}$$

where

$$M(z) = (I - zA_1^{-1}B_1)^{-1}(zA_1^{-1}B_0 - A_1^{-1}A_0) \tag{21}$$

is the amplification matrix. If as *z* tends to infinity (21) tends to zero (i.e. $M(\infty) = 0$), it means that an A-stable (11) is *L*-stable. Then the stability function $P(z)$ is

$$P(z) = Det[I_k R - M(z)] = R^{k-1}(R - D(z)) \tag{22}$$

The stability domain *S* of the method is

$$S = \{z \in \mathrm{C} : |R(z)| \leq 1\} \tag{23}$$

The $D(z)$ (the only non-zero value of *R*(*z*)) for the proposed methods are given as a rational function $D(z) = \frac{P(z)}{Q(z)}$, where P (z) and Q(z) are polynomials.

RESULTS AND DISCUSSION

Numerical Experiment

Problem 1: Consider the test equation with $y'' = -10y,\ y(0) = 0,\ y'(1) = 1$ exact solution $y(x) = e^{-10x}$, h = 0.02 using the developed method $p = 3,\ 5,\ \&\ 7$ and the results displayed in Table 1.

Table 1: Error of Derived Methods for $p = 3,\ 5,\ \&\ 7$

h	Error in $p=3$	Error in $p=5$	Error in $p=7$
0.02	1.40E-008	1.96E-008	8.78E-010
0.04	1.44E-008	1.65E-008	8.53E-010
0.06	-7.59E-008	2.01E-008	7.56E-010
0.08	1.41E-009	3.62E-009	-8.11E-010
0.10	2.92E-009	3.19E-009	-4.20E-010
0.12	-9.67E-010	2.88E-009	-2.11E-011
0.14	-3.28E-010	2.87E-010	-1.68E-011

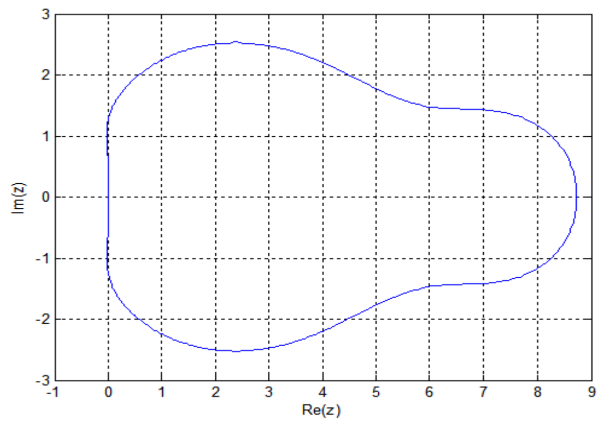

Figure 1. Region of Absolute Stability for $p=3$

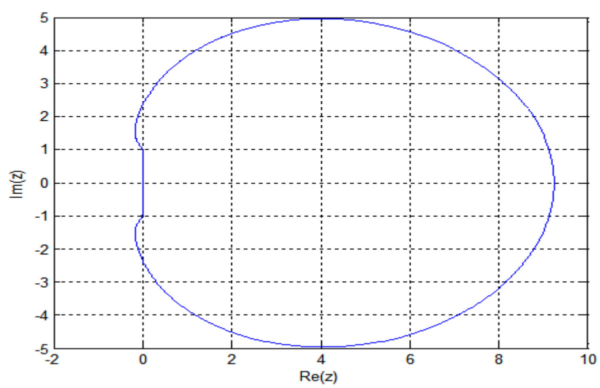

Figure 2. Region of Absolute Stability for $p=5$

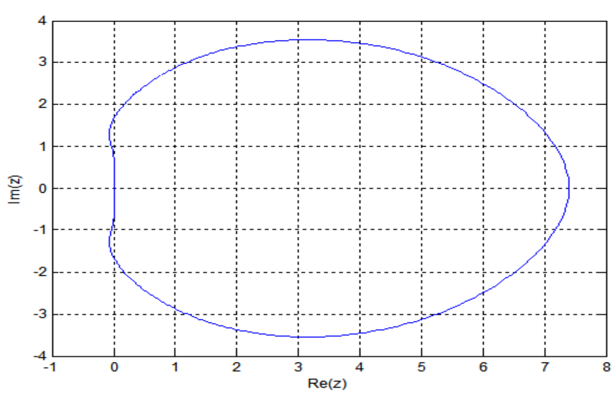

Figure 3. Region of Absolute Stability for $p=7$

Problem 2:

$$y'' = \begin{pmatrix} -21 & 19 & -20 \\ 19 & -21 & 20 \\ 40 & -40 & -40 \end{pmatrix} y; \quad y(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}; \; y'(0) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

The theoretical solution of the problem is:

$$y(t) = \frac{1}{2}\begin{pmatrix} e^{-2t} + e^{-40t}(\cos(40t) + \sin(40t)) \\ e^{-2t} - e^{-40t}(\cos(40t) + \sin(40t)) \\ 2e^{-40t}(\sin(40t) - \cos(40t)) \end{pmatrix}$$

Table 2. Error of Derived Methods for $p = 3,\ 5,\ \&\ 7$

h	***Error p = 3***	***Error p = 5***	***Error p =7***
0.01	2.15E-02	4.93E-03	2.18E-03
0.005	3.97E-03	3.11E-04	4.42E-04
0.0025	5.53E-05	5.20E-05	4.17E-06
0.00125	7.19E-05	1.45E-06	5.03E-08
0.000625	2.28E-06	3.06E-07	1.26E-09

CONCLUSION

Self-starting block backward differentiation methods using multistep have been developed. The result has shown that it is zero stable for all $k \geq 3$, *l*-stable for $k \leq 7$. The methods are convergent and absolutely stable. Two numerical problems, stiff and non-stiff initial value problems were tested for $p = 3,\ 5,\ \&\ 7$. The computational results reveal that the new block methods work well with good error stability.

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