



The Out-of-Plane Libration Points in ER3BP With Radiation of The Smaller Primary, Triaxiality of Both Primaries Surrounded by a BELT

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ABSTRACT

The paper investigates the motion of an infinitesimal particle near the out-of-plane equilibrium points in the elliptic restricted three body problem (ER3BP) when the primaries are triaxial rigid bodies with the smaller primary a source of radiation and surrounded by a belt. It is observed that there exist two out-of-plane equilibria which lie in the $\xi\xi$ - plane in symmetrical positions with respect to the orbital plane. The parameters involved in the system affect their positions. The position changes with an increase in triaxiality, radiation and belt. The position and stability of the out-of-plane equilibrium points are investigated numerically using two binary systems (Xi-Bootis and Kruger 60) and they are found to be unstable.

INTRODUCTION

In celestial mechanics the three-body problem is one of the most important areas of research, most especially in the field of astrodynamics and astrophysics. Renowned mathematicians and scientists have obtained interesting and significant results in an attempt to understand and predict the motion of natural bodies.

The restricted three-body problem is a configuration involving two massive bodies called the primaries and a particle of negligible mass, called the third body (infinitesimal particle, test particle). It explains the motion of the infinitesimal particle in the vicinity of the primaries which move in circular or elliptic orbits around their common centre of mass due to their mutual gravitational attraction. It possesses three collinear points $L_{1,2,3}$ and two triangular points $L_{4,5}$. They lie on the orbital plane of motion of the primaries. The latter are conditionally stable, while the former are unstable.

In the classical restricted three-body problem (CR3BP) only gravitational forces influence the motion of the particle. The photogravitational restricted three-body problem (R3BP) problem arises when one of the participating bodies or both are intense emitters of radiation. It is inadequate to consider only the gravitational force in some solar or stellar dynamic problems. For instance, gravity is not only the dominant force present when a star collides with a particle, but also the repulsive forces of radiation pressure (Radviesky, 1950). Due to the importance of this force, several authors (Chermikov, 1970; Simmons et.al. 1985; Das et.al.2009;

Singh and Umar 2012; Abouelmagd, 2013; Ammar 2008) studied its influence on the stability of infinitesimal mass in their papers.

In addition to this force, many authors have extended the work to include the role played by the shape of bodies (triaxiality, oblateness and so on). These two Pertubing forces (triaxiality and oblateness) have been studied extensively by researchers in the R3BP.

The first to point out the existence of out-of-plane equilibrium points was (Radvieskii, 1950) who studied the case of sun planet-particle and Galaxy-kernel-sun-particle and found the two equilibrium points $L_{6,7}$ on the $\xi\xi$ -plane to be symmetrical with respect to the orbital plane.

Doukos and Markellos (2006) determined the out-of-plane points numerically by approximation with power series expansion. They showed that the out-of-plane points exist, but they are unstable. A generalized out-of-plane model was studied by Hussain and Umar (2019) in which the primary is oblate and the secondary is triaxial and radiating in the ER3BP, it is observed that the out-of-plane equilibrium points $L_{6,7}$ area affected by the oblateness of the primary, radiation pressure and triaxiality of the secondary, semi-major axis and eccentricity.

It was also established by Singh and Umar (2013) that the position and stability of out-of-plane points in ER3BP are greatly affected by oblateness and radiation pressure of the primaries and the eccentricity of their orbits.

Singh and Tyokyaa (2021) examined the motion of a test particle around the out-of-plane libration points in the ER3BP with oblateness up to zonal harmonics J_4 . It is found that their positions are affected by the oblateness of the primaries, the semi-major axis and eccentricity of their orbits.

Several authors such as (Singh, 2012; Singh and Amuda 2015; Naraya and Nutan, 2014; Zotos 2018) have studied the out-of-plane points under the influence of radiation pressure or PR-drag or oblateness or in combination of one of these forces and they all found the out-of-plane equilibrium points unstable in the CR3BP or ER3BP. The basins of attraction around the out-of-plane points in the Copenhagen R3BP was determined by Zotos (2018) using a multivariate version of the Newton-Raphson interactive method.

The discovery of many extra solar planetary systems revealed the presence of belts of dust particles that are regarded as the young analogues of Kuiper belt. (Aumman et al. 1984) and Jiang & Yeh (2003) studied the effects of belts on planetary orbits and observed that the planets might prefer to stay near the inner part instead of outer part of the belt. Later the R3BP was modified in their paper Jiang & Yeh (2004) to include the effect of additional gravitational force from the belt on the infinitesimal mass, which results in the formation of new libration points.

Singh and Taura (2014b) studied the model of restricted three-body problem when the two primaries are oblate spheroids and radiating with the gravitational potential from a belt. They obtained in addition to the usual five libration points two new collinear points as a result of the potential from the belt. The influence of the belt and non-sphericity of the primaries on the infinitesimal mass was studied by Singh and Taura (2014c). They did analytic and numerical treatment of motion of a dust grain particle around triangular equilibrium points when the bigger primary is triaxial and the smaller one an oblate spheroid with a potential from the belt. They found that triangular points are stable for $0 < \mu < \mu_c$ and unstable for $\mu_c \leq \mu \leq \frac{1}{2}$, where μ_c is the critical mass ratio. It was also observed that the potential from the belt increase the range of stability. Singh and Tyokyaa (2023) studied the collinear investigation of albedo effects in the elliptic restricted three-body problem with an oblate primary, a triaxial secondary and a potential due to belt. The result revealed that, as the half distance between the mass dipoles and oblateness of the primary increase, the region of stability of the collinear positions decreases for the aforementioned binary systems.

The case when the more massive primary is a triaxial body and less massive one an oblate spheroid emitting radiation enclosed by a circumbinary disc (belt) in the presence of Poyting-Robertson drag (P-R Drag) force was investigated by Singh and Amuda (2019). They observed

that the potential from the belt is a stabilizing force as it can change an unstable condition to a stable one even when the mass parameter exceeds the critical mass value ($\mu > \mu_c$).

In this paper our aim is to investigate the effect of triaxiality and radiation pressure of the smaller primary together with the potential of the belt enclosing them on the stability of a test particle around the out-of-plane equilibrium points in the ER3BP.

MATERIALS AND METHODS

Location and Linear stability of out-of-plane triangular equilibrium points

The equation of motion of an infinitesimal particle in the ER3BP when the primaries are triaxial and smaller one radiating, with a gravitational potential from a belt, in a dimensionless rotating coordinate system (ξ, η, ζ) are as follows:

$$\begin{aligned} \xi'' - 2\eta' &= \Omega_\xi \\ \eta'' + 2\xi' &= \Omega_\eta \\ \zeta'' &= \Omega_\zeta \end{aligned} \tag{1}$$

$$\begin{aligned} \Omega = (1 - e^2)^{-1/2} & \left[\frac{1}{2}(\xi^2 + \eta^2) + \frac{1}{n^2} \left\{ \frac{(1-\mu)}{r_1} + \right. \right. \\ & \frac{(1-\mu)(2\sigma_1 - \sigma_2)}{2r_1^3} - \frac{3(1-\mu)(\sigma_1 - \sigma_2)\eta^2}{2r_1^5} - \frac{3(1-\mu)\sigma_1\zeta^2}{2r_1^5} + \frac{\mu q_2}{r_2} + \\ & \left. \left. \frac{\mu(2\sigma_3 - \sigma_4)q_2}{2r_2^3} - \frac{3\mu(\sigma_3 - \sigma_4)q_2\eta^2}{2r_2^5} - \frac{3\mu\sigma_3 q_2 \zeta^2}{2r_2^5} + \right. \right. \\ & \left. \left. \frac{M_b}{[r^2 + (c + \sqrt{\xi^2 + d^2})^2]^{1/2}} \right\} \right] \end{aligned} \tag{2}$$

$$\begin{aligned} r_1^2 &= (\xi + \mu)^2 + \eta^2 + \zeta^2 \\ r_2^2 &= (\xi + \mu - 1)^2 + \eta^2 + \zeta^2 \end{aligned} \tag{3}$$

$$\begin{aligned} n^2 &= \frac{1}{a} \left[1 + \frac{3}{2}e^2 + \frac{3}{2}(2\sigma_1 - \sigma_2) + \frac{3}{2}(2\sigma_3 - \sigma_4) + \right. \\ & \left. \frac{2M_b r_c}{[r_c^2 + T^2]^{3/2}} \right] \end{aligned} \tag{4}$$

r is the radial distance of the infinitesimal mass and is given by $r^2 = \xi^2 + \zeta^2$, where c and d are the parameters which determine the density profile of the belt Miyamoto and Nagai (1975) and (Kushvah, 2008) r_c is the distance of any out-of-plane point from the origin and T is their sum, r_1 and r_2 are distances of the bigger and smaller primaries from the infinitesimal particle, respectively. q_1 and q_2 are their mass reduction factor (radiation factor), while (σ_1, σ_2) and (σ_3, σ_4) denote their triaxiality, respectively. n is the mean motion, a and e are the semi major axis and the eccentricity of the elliptic orbit respectively.

The equilibrium points are the solutions of the system of

$$\Omega_{\xi} = \Omega_{\eta} = \Omega_{\zeta} = 0, i. e. \left\{ \begin{aligned} &\xi - \frac{1}{n^2} \left[\frac{(1-\mu)(\xi+\mu)}{r_1^3} + \frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2r_1^5} \eta^2 - \frac{15(1-\mu)(\xi+\mu)\sigma_1\zeta^2}{2r_1^7} + \right. \\ &\frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2r_1^5} - \frac{15(1-\mu)(\xi+\mu)\sigma_1}{2r_1^7} \eta^2 - \frac{15(1-\mu)(\xi+\mu)\sigma_1\zeta^2}{2r_1^7} + \\ &\frac{\mu(\xi+\mu-1)q_2}{r_2^3} + \frac{3\mu(\xi+\mu-1)(2\sigma_3-\sigma_4)q_2}{2r_2^5} - \frac{15\mu(\xi+\mu-1)\sigma_3}{2r_2^7} q_2 \eta^2 - \\ &\left. \frac{15\mu(\xi+\mu-1)\sigma_3 q_2 \zeta^2}{2r_2^7} + \frac{M_b \xi}{[\xi^2+(c+\sqrt{\zeta^2+d^2})^2]^{3/2}} \right] \Bigg\} = 0 \end{aligned} \right. \quad (5)$$

$$\Omega_{\eta} = (1 - e^2)^{-1/2} \eta \left\{ 1 - \frac{1}{n^2} \left[\frac{(1-\mu)}{r_1^3} + \frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2r_1^5} + \frac{3(1-\mu)(\sigma_1-\sigma_2)}{r_1^5} - \frac{15(1-\mu)(\sigma_1-\sigma_2)}{2r_1^7} \eta^2 - \frac{15(1-\mu)\sigma_1\zeta^2}{2r_1^7} + \frac{\mu q_2}{r_2^3} + \frac{3\mu(2\sigma_3-\sigma_4)q_2}{2r_2^5} + \frac{3\mu(\sigma_3-\sigma_4)}{r_2^5} q_2 - \frac{15\mu(\sigma_3-\sigma_4)}{2r_2^7} q_2 \eta^2 - \frac{15\mu\sigma_3 q_2 \zeta^2}{2r_2^7} + \frac{M_b}{[\xi^2+(c+\sqrt{\zeta^2+d^2})^2]^{3/2}} \right] \right\} = 0 \quad (6)$$

$$\Omega_{\zeta} = (1 - e^2)^{-1/2} \left[-\frac{\zeta}{n^2} \left[\frac{(1-\mu)}{r_1^3} + \frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2r_1^5} + \frac{3(1-\mu)\sigma_1}{r_1^5} - \frac{15(1-\mu)(\sigma_1-\sigma_2)}{2r_1^7} \eta^2 - \frac{15(1-\mu)\sigma_1\zeta^2}{2r_1^7} + \frac{\mu q_2}{r_2^3} + \frac{3\mu(2\sigma_3-\sigma_4)}{2r_2^5} q_2 + \frac{3\mu\sigma_3}{r_2^5} q_2 - \frac{15\mu(\sigma_3-\sigma_4)}{2r_2^7} q_2 \eta^2 - \frac{15\mu\sigma_3 q_2 \zeta^2}{2r_2^7} + \frac{M_b [c(\zeta^2+d^2)^{-1/2}+1]}{[\xi^2+(c+\sqrt{\zeta^2+d^2})^2]^{3/2}} \right] \right] = 0 \quad (7)$$

The out-of-plane equilibrium points are the solution of above equations, when

$$\xi \neq 0, \quad \eta = 0 \quad \text{and} \quad \zeta \neq 0$$

From (7) with $\zeta \neq 0$ we get:

$$\frac{(1-\mu)}{r_1^3} + \frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2r_1^5} + \frac{3(1-\mu)\sigma_1}{r_1^5} - \frac{15(1-\mu)\zeta^2}{2r_1^7} + \frac{\mu q_2}{r_2^3} + \frac{3\mu(2\sigma_3-\sigma_4)}{2r_2^5} q_2 + \frac{3\mu\sigma_3}{r_2^5} q_2 - \frac{15\mu\sigma_3 q_2 \zeta^2}{2r_2^7} + \frac{M_b [c(\zeta^2+d^2)^{-1/2}+1]}{[\xi^2+(c+\sqrt{\zeta^2+d^2})^2]^{3/2}} = 0 \quad (8)$$

Let $Q_1 = (1 - \mu)$ and $Q_2 = \mu q_2$, then (8) becomes:

$$\frac{Q_1}{r_1^3} + \frac{3Q_1(2\sigma_1-\sigma_2)}{2r_1^5} + \frac{3Q_1\sigma_1}{r_1^5} - \frac{15Q_1\sigma_1\zeta^2}{2r_1^7} + \frac{Q_2}{r_2^3} + \frac{3Q_2(2\sigma_3-\sigma_4)}{2r_2^5} + \frac{3Q_2\sigma_3}{r_2^5} - \frac{15Q_2\sigma_3\zeta^2}{2r_2^7} + \frac{M_b [c(\zeta^2+d^2)^{-1/2}+1]}{[\xi^2+(c+\sqrt{\zeta^2+d^2})^2]^{3/2}} = 0 \quad (9)$$

Also from (5) we write:

$$n^2 \xi - \frac{Q_1(\xi+\mu)}{r_1^3} - \frac{3Q_1(\xi+\mu)(2\sigma_1-\sigma_2)}{2r_1^5} + \frac{15Q_1(\xi+\mu)\sigma_1\zeta^2}{2r_1^7} - \frac{Q_2(\xi+\mu-1)}{r_2^3} - \frac{3Q_2(\xi+\mu-1)(2\sigma_3-\sigma_4)}{2r_2^5} + \frac{15Q_2(\xi+\mu-1)\sigma_3\zeta^2}{2r_2^7} - \frac{M_b \xi}{[\xi^2+(c+\sqrt{\zeta^2+d^2})^2]^{3/2}} = 0 \quad (10)$$

Expanding (10) we obtained:

$$\xi \left\{ 1 - \frac{1}{n^2} \left(\frac{Q_1}{r_1^3} + \frac{3Q_1(2\sigma_1-\sigma_2)}{2r_1^5} - \frac{15Q_1\sigma_1\zeta^2}{2r_1^7} + \frac{Q_2}{r_2^3} + \frac{3Q_2(2\sigma_3-\sigma_4)}{2r_2^5} - \frac{15Q_2\sigma_3\zeta^2}{2r_2^7} + \frac{M_b}{[\xi^2+(c+\sqrt{\zeta^2+d^2})^2]^{3/2}} \right) \right\} - \frac{\mu}{n^2} \left(\frac{Q_1}{r_1^3} - \frac{15Q_1\sigma_1\zeta^2}{2r_1^7} + \frac{Q_2}{r_2^3} - \frac{15Q_2\sigma_3\zeta^2}{2r_2^7} + \frac{3Q_1(2\sigma_1-\sigma_2)}{2r_1^5} + \frac{3Q_2(2\sigma_3-\sigma_4)}{2r_2^5} \right) + \frac{1}{n^2} \left(\frac{Q_2}{r_2^3} + \frac{3Q_2(2\sigma_3-\sigma_4)}{2r_2^5} - \frac{15Q_2\sigma_3\zeta^2}{2r_2^7} \right) = 0 \quad (11)$$

From (9) we have

$$\frac{15Q_1(\sigma_1 - \sigma_2)\zeta^2}{2r_1^7} + \frac{15Q_2(\sigma_3 - \sigma_4)\zeta^2}{2r_2^7} = \frac{Q_1}{r_1^3} + \frac{3Q_1(2\sigma_1 - \sigma_2)}{2r_1^5} + \frac{3Q_1\sigma_1}{r_1^5} + \frac{Q_2}{r_2^3} + \frac{3Q_2(2\sigma_3 - \sigma_4)}{2r_2^5} + \frac{3Q_2\sigma_3}{r_2^5} + \frac{M_b [c(\zeta^2+d^2)^{-1/2}+1]}{[\xi^2+(c+\sqrt{\zeta^2+d^2})^2]^{3/2}}$$

$$\zeta^2 = \frac{2r_1^7 r_2^7}{15Q_1(\sigma_1 - \sigma_2) r_2^7 + 15Q_2(2\sigma_3 - \sigma_4) r_1^7} \left\{ \frac{Q_1}{r_1^3} + \frac{3Q_1(2\sigma_1 - \sigma_2)}{2r_1^5} + \frac{3Q_1\sigma_1}{r_1^5} + \frac{Q_2}{r_2^3} + \frac{3Q_2(2\sigma_3 - \sigma_4)}{2r_2^5} + \frac{3Q_2\sigma_3}{r_2^5} + \frac{M_b [c(\zeta^2+d^2)^{-1/2}+1]}{[\xi^2+(c+\sqrt{\zeta^2+d^2})^2]^{3/2}} \right\} \quad (12)$$

Substituting (9) into (11) and solving we obtained:

$$\xi \left\{ 1 - \frac{1}{n^2} \left(-\frac{3Q_1\sigma_1}{r_1^5} - \frac{3Q_2\sigma_3}{r_2^5} + \frac{M_b}{[\xi^2+(c+\sqrt{\zeta^2+d^2})^2]^{3/2}} - \frac{M_b [c(\zeta^2+d^2)^{-1/2}+1]}{[\xi^2+(c+\sqrt{\zeta^2+d^2})^2]^{3/2}} \right) \right\} - \frac{\mu}{n^2} \left(-\frac{3Q_1\sigma_1}{r_1^5} - \frac{3Q_2\sigma_3}{r_2^5} - \frac{M_b [c(\zeta^2+d^2)^{-1/2}+1]}{[\xi^2+(c+\sqrt{\zeta^2+d^2})^2]^{3/2}} \right)$$

$$\begin{aligned}
 & + \frac{1}{n^2} \left(-\frac{Q_1}{r_1^3} - \frac{3Q_1(2\sigma_1 - \sigma_2)}{2r_1^5} - \frac{3Q_1\sigma_1}{r_1^5} - \frac{3Q_2\sigma_3}{r_2^5} + \right. \\
 & \left. \frac{15Q_1\sigma_1\zeta^2}{2r_1^7} - \frac{M_b [c + (\zeta^2 + d^2)^{-1/2 + 1}]}{[\xi^2 + (c + \sqrt{\zeta^2 + d^2})^2]^{3/2}} \right) = 0 \\
 \xi = & \frac{\frac{Q_1}{r_1^3} + \frac{3Q_1(2\sigma_1 - \sigma_2)}{2r_1^5} + \frac{3Q_1\sigma_1(1-\mu)}{r_1^5} + \frac{3Q_2Q_1\sigma_3}{r_2^5} - \frac{15Q_1\sigma_1\zeta^2}{2r_1^7} +}{\frac{M_b Q_1 [c + (\zeta^2 + d^2)^{-1/2 + 1}]}{[\xi^2 + (c + \sqrt{\zeta^2 + d^2})^2]^{3/2}}} \\
 & \frac{n^2 + \frac{3Q_1\sigma_1}{r_1^5} + \frac{3Q_2\sigma_3}{r_2^5} + \frac{M_b}{[\xi^2 + (c + \sqrt{\zeta^2 + d^2})^2]^{3/2}} + \frac{M_b [c(\zeta^2 + d^2)^{-1/2 + 1}]}{[\xi^2 + (c + \sqrt{\zeta^2 + d^2})^2]^{3/2}}}{(1-\mu) \left\{ \frac{\frac{1}{r_1^3} + \frac{3(2\sigma_1 - \sigma_2)}{2r_1^5} + \frac{3Q_1\sigma_1}{r_1^5} + \frac{3Q_2\sigma_3}{r_2^5} - \frac{15\sigma_1\zeta^2}{2r_1^7} +}{\frac{M_b [c + (\zeta^2 + d^2)^{-1/2 + 1}]}{[\xi^2 + (c + \sqrt{\zeta^2 + d^2})^2]^{3/2}}} \right\}} \\
 & \frac{n^2 + \frac{3Q_1\sigma_1}{r_1^5} + \frac{3Q_2\sigma_3}{r_2^5} + \frac{M_b}{[\xi^2 + (c + \sqrt{\zeta^2 + d^2})^2]^{3/2}} + \frac{M_b [c(\zeta^2 + d^2)^{-1/2 + 1}]}{[\xi^2 + (c + \sqrt{\zeta^2 + d^2})^2]^{3/2}}}{(13)}
 \end{aligned}$$

We use the initial approximation $\xi_o = (1 - \mu)$ and $\zeta_o = \sqrt{3(2\sigma_3 - \sigma_4)}$ to obtain the positions of out-of-plane equilibrium points $L_{6,7}$ with the aid of the software package MATHEMATICA 10.4 in the form of power series to third order term in $(2\sigma_3 - \sigma_4)$ from (12) and (13) as:

$$\begin{aligned}
 \xi_o = & \frac{(-1+\mu) - 3\sqrt{3}(-1+\mu)(2-3e^2-2a+(2\sigma_2-\sigma_1)(3-3a) \times (2\sigma_3-\sigma_4)^{3/2}}{2(a\mu q_2)} \\
 & - 9(\sqrt{3}(1+\mu)(2+3a(2+15(2\sigma_2-\sigma_1))(2\sigma_3-\sigma_4)^{5/2} - 27(-1+\mu)(2+3e^2-2a) \\
 & \frac{4(a\mu q_2)}{+ (2\sigma_1 - \sigma_2)(-3 + 6a)(-1 + \mu)q_2)}(4(\sigma^2\mu^2q_2^2)^{-1}(2\sigma_3 - \sigma_4)^3 + 0(2\sigma_3 - \sigma_4)^{7/2} \\
 & (14)
 \end{aligned}$$

$$\begin{aligned}
 \zeta_o = & \sqrt{3}\sqrt{(2\sigma_3 - \sigma_4)} - \frac{9(-1+\mu)(2+9(2\sigma_1-\sigma_2))}{10\mu q_2}(2\sigma_3 - \sigma_4)^2 + \frac{81(-1+\mu)}{20\mu q_2} \\
 & (2\sigma_1 - \sigma_2)q_2(2\sigma_3 - \sigma_4)^3 - 0(2\sigma_3 - \sigma_4)^{7/2} \quad (15)
 \end{aligned}$$

The equilibrium points $(\xi_o, 0, \pm\zeta_o)$ given by equations (14) and (15) are called the out-of-plane equilibrium points and are denoted by L_6 and L_7 respectively.

Linear stability of out-of-plane equilibrium points

The stability of these equilibrium points is determined by the eigen-values of the characteristic equation (16). If the roots of the characteristic equation are pure imaginary

roots or complex roots with negative real parts the equilibrium point will be stable otherwise it will be unstable. Near any one of the out-of-plane equilibrium points the characteristic equation of the system can be written as:

$$\begin{aligned}
 & \lambda^6 + (4 - \Omega^0_{\xi\xi} - \Omega^0_{\eta\eta} - \Omega^0_{\zeta\zeta})\lambda^4 + (\Omega^0_{\eta\eta}\Omega^0_{\zeta\zeta} + \Omega^0_{\xi\xi}\Omega^0_{\zeta\zeta} + \Omega^0_{\xi\xi}\Omega^0_{\eta\eta} - 4\Omega^0_{\zeta\zeta} - (\Omega^0_{\xi\xi})^2)\lambda^2 - (\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}\Omega^0_{\zeta\zeta} - (\Omega^0_{\xi\xi})^2\Omega^0_{\eta\eta}) = 0 \quad (16)
 \end{aligned}$$

The superscript O denotes that the partial derivatives are evaluated at the out-of-plane point (ξ_o, o, ζ_o) where we have:

$$\begin{aligned}
 \Omega^0_{\xi\xi} = & (1 - e^2)^{-1/2} \left[1 + \frac{1}{n^2} \left\{ \frac{3Q_1(\xi_o + \mu)^2}{r_{10}^3} - \frac{Q_1}{r_{10}^3} + \frac{15Q_1(\xi_o + \mu)^2(2\sigma_1 - \sigma_2)}{2r_{10}^7} - \frac{3Q_1}{2r_{10}^5} + \frac{105Q_1(\xi_o + \mu)^2\sigma_1\zeta_o^2}{2r_{10}^9} - \frac{15Q_1\sigma_1\zeta_o^2}{2r_{10}^7} + \frac{3Q_2(\xi_o + \mu - 1)^2}{r_{20}^5} - \frac{Q_2}{r_{20}^5} + \frac{15Q_2(\xi_o + \mu - 1)^2(2\sigma_3 - \sigma_4)}{2r_{20}^7} - \frac{3Q_2}{2r_{20}^5} - \frac{105Q_2(\xi_o + \mu - 1)^2\sigma_3\zeta_o^2}{2r_{20}^9} + \frac{15Q_2\sigma_3\zeta_o^2}{2r_{20}^7} + \frac{3M_b\xi_o^2}{\left[\xi_o^2 + (c + \sqrt{\zeta_o^2 + d^2})^2 \right]^{5/2}} - \frac{M_b}{\left[\xi_o^2 + (c + \sqrt{\zeta_o^2 + d^2})^2 \right]^{3/2}} \right\} \right] \quad (17)
 \end{aligned}$$

$$\begin{aligned}
 \Omega^0_{\eta\eta} = & (1 - e^2)^{-1/2} \left[1 - \frac{1}{n^2} \left\{ \frac{Q_1}{r_{10}^3} + \frac{3Q_1(2\sigma_1 - \sigma_2)}{2r_{10}^5} - \frac{15Q_1\sigma_1\zeta_o^2}{2r_{10}^7} + \frac{Q_2}{r_{20}^3} + \frac{3Q_2(2\sigma_3 - \sigma_4)}{2r_{20}^5} - \frac{15Q_2\sigma_3\zeta_o^2}{2r_{20}^7} - \frac{3M_b\eta_o^2}{\left[\xi_o^2 + (c + \sqrt{\zeta_o^2 + d^2})^2 \right]^{5/2}} + \frac{M_b}{\left[\xi_o^2 + (c + \sqrt{\zeta_o^2 + d^2})^2 \right]^{3/2}} \right\} \right] \quad (18)
 \end{aligned}$$

$$\begin{aligned}
 \Omega^0_{\zeta\zeta} = & (1 - e^2)^{-1/2} \left[\frac{1}{n^2} \left\{ -\frac{Q_1}{r_{10}^3} + \frac{3Q_1\zeta_o^2}{r_{10}^5} - \frac{3Q_1(2\sigma_1 - \sigma_2)}{2r_{10}^5} + \frac{15Q_1(2\sigma_1 - \sigma_2)\zeta_o^2}{2r_{10}^7} - \frac{3Q_1\sigma_1}{r_{10}^5} + \frac{15Q_1\sigma_1\zeta_o^2}{r_{10}^7} + \frac{45Q_1\sigma_1\zeta_o^2}{2r_{10}^7} - \frac{105Q_1\sigma_1\zeta_o^4}{2r_{10}^9} - \frac{Q_2}{r_{20}^3} + \frac{3Q_2\zeta_o^2}{r_{20}^5} - \frac{3Q_2(2\sigma_3 - \sigma_4)}{2r_{20}^5} + \frac{15Q_2(2\sigma_3 - \sigma_4)\zeta_o^2}{2r_{20}^7} - \frac{3Q_2\sigma_3}{r_{20}^5} + \frac{15Q_2\sigma_3\zeta_o^2}{2r_{20}^7} + \frac{45Q_2\sigma_3\zeta_o^2}{2r_{20}^7} - \frac{M_b [c(\zeta_o^2 + d^2)^{-1/2 + 1}]}{2r_{20}^9} - \frac{M_b}{\left[\xi_o^2 + (c + \sqrt{\zeta_o^2 + d^2})^2 \right]^{3/2}} + \frac{M_b c^2 \zeta_o^2 [(\zeta_o^2 + d^2)^{-3/2}]}{\left[\xi_o^2 + (c + \sqrt{\zeta_o^2 + d^2})^2 \right]^{3/2}} + \frac{3M_b \zeta_o^2 [c(\zeta_o^2 + d^2)^{-1/2 + 1}]}{\left[\xi_o^2 + (c + \sqrt{\zeta_o^2 + d^2})^2 \right]^{5/2}} \right\} \right] \quad (19)
 \end{aligned}$$

$$\Omega^0_{\xi\zeta} = (1 - e^2)^{-1/2} \left[\frac{3\zeta_0}{n^2} \left\{ \frac{Q_1(\xi_0 + \mu)}{r_{10}^5} + \frac{5Q_1(\xi_0 + \mu)(2\sigma_1 - \sigma_2)}{2r_{10}^7} - \frac{35Q_1(\xi_0 + \mu)\sigma_1\zeta_0^2}{2r_{10}^9} - \frac{15Q_1\sigma_1\zeta_0^2}{r_{10}^7} + \frac{Q_2(\xi_0 + \mu - 1)}{r_{20}^5} + \frac{5Q_2(\xi_0 + \mu - 1)(2\sigma_3 - \sigma_4)}{2r_{20}^7} - \frac{35Q_2(\xi_0 + \mu - 1)\sigma_3\zeta_0^2}{2r_{20}^9} + \frac{15Q_2\sigma_3\zeta_0^2}{r_{20}^7} + \frac{3M_b\xi_0^2 \left[c(\zeta_0^2 + d^2)^{-1/2} + 1 \right]}{\left[\xi_0^2 + \left(c + \sqrt{\zeta_0^2 + d^2} \right)^2 \right]^{5/2}} \right\} \right] \quad (20)$$

RESULTS AND DISCUSSION

The effect of triaxiality, belt and radiation pressure on the locations and stability of out-of-plane equilibrium points are analyzed using the values of the parameters of the binary system (xi-Bootis and Kruger-60) with the aid of mathematical software MATHEMATICA 10.4 and are presented in Table 2- 6.

The numerical data of the binary system xi-Bootis and Kruger-60 are presented below:

Table 1: Numerical data for the Binary System

Binary system	Masses (M \odot)		Eccentricity (e)	Semi-major axis (a)	Luminosity L \odot		Spectral Types
	M ₁	M ₂			L ₁	L ₂	
Xi Bootis	0.9	0.66	0.5117	4.9044	0.49	0.061	G8/k4
Kruger 60	0.271	0.176	0.4100	2.3830	0.01	0.0034	M3/M4

Source:NASA ADS

Table 2: The effect of triaxiality on the location and stability of out-of-plane equilibrium points of xi-Bootis for e = 0.5117, a = 0.7304, $\mu = 0.4231$, $q_2 = 0.9998$.

Triaxiality				Out-of-Plane points		Roots of the characteristic equation		
σ_1	σ_2	σ_3	σ_4	ξ	$\pm\zeta$	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
0.015	0.011	0.002	0.001	0.466009	0.265523	± 610.224	$-173.012 \pm 175.323i$	$173.012 \pm 175.323i$
0.02	0.015	0.003	0.002	0.477622	0.257321	± 814.001	$-172.807 \pm 177.887i$	$172.807 \pm 177.887i$
0.03	0.019	0.004	0.003	0.49838	0.246711	± 998.013	$-169.557 \pm 178.39i$	$169.557 \pm 178.39i$
0.04	0.02	0.005	0.004	0.526424	0.206453	± 1522.48	$-165.09 \pm 179.832i$	$165.09 \pm 179.832i$
0.05	0.03	0.006	0.005	0.528303	0.204726	± 1321.43	$-162.88 \pm 180.378i$	$162.88 \pm 180.378i$

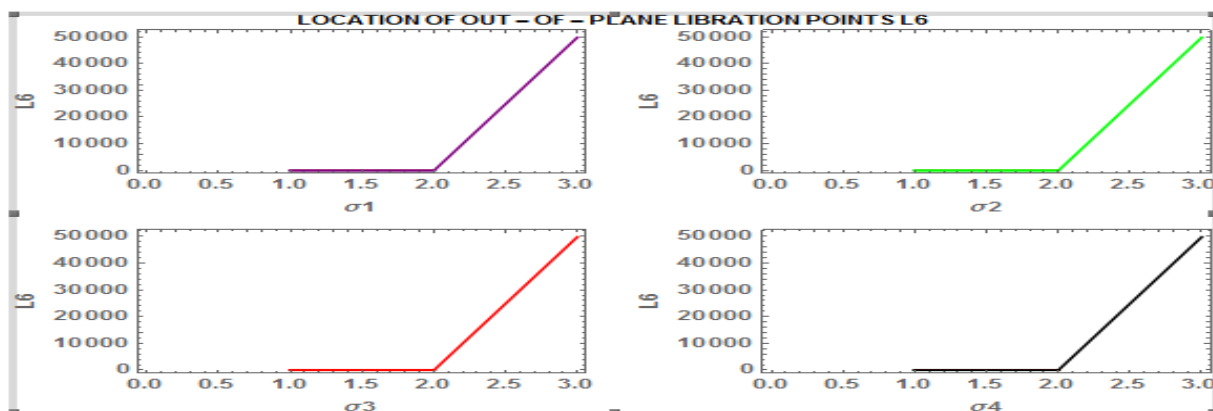


Figure 1: Effect of triaxiality on the Out-of-Plane Libration points (L_6)

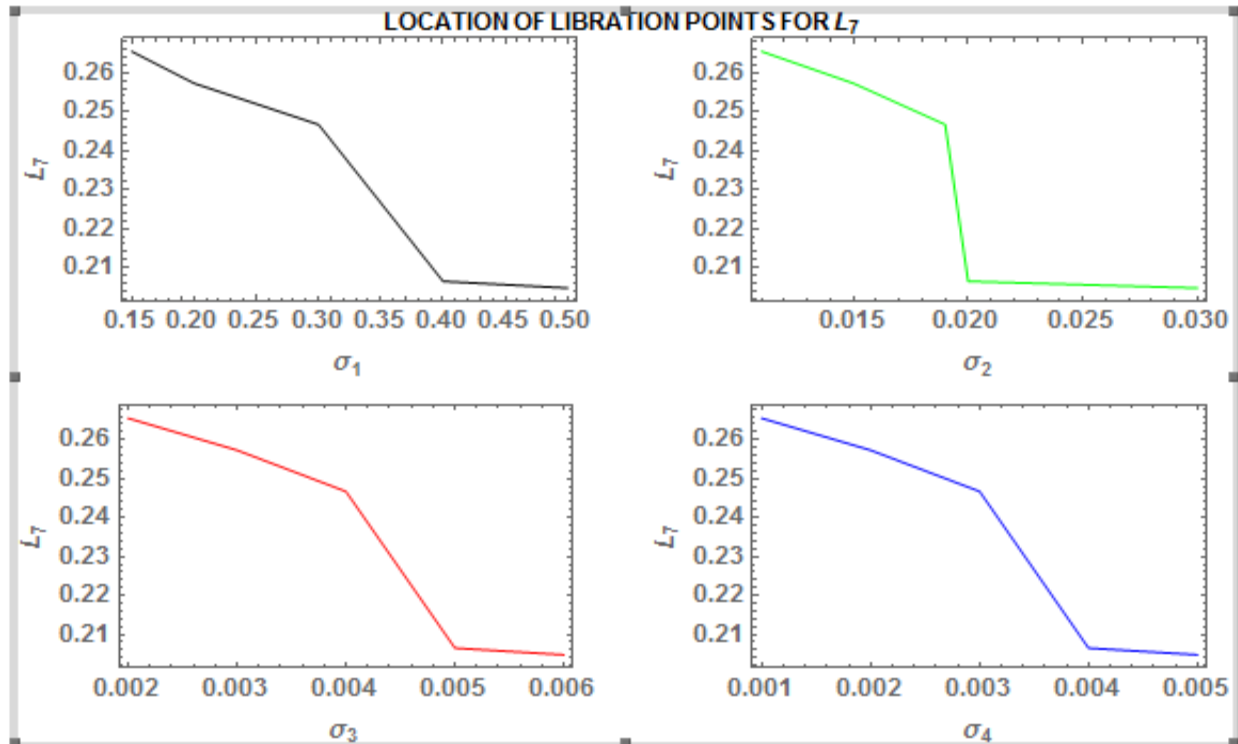


Figure 2: Effect of triaxiality on the Out-of-Plane Libration points (L_7).

Table 3: The effect of belt (M_b) on the location and stability of out-of-plane equilibrium points of xi-Bootis for $e = 0.5117, a = 0.7304, \mu = 0.4231, q_1 = 0.9988, q_2 = 0.9998, \sigma_1=0.02, \sigma_2=0.015, \sigma_3=0.003, \sigma_4=0.002$

M_b	Out-of plane points		Roots of the characteristic equation		
	ξ	$\pm\zeta$	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
0.02	0.520081	0.211443	± 47.114534	$\pm 113.35634i$	± 33.3587
0.03	0.512573	0.211498	± 47.244377	$\pm 120.48473i$	± 34.33589
0.04	0.510623	0.221980	± 48.257841	$\pm 120.67357i$	± 34.367635
0.05	0.510327	0.222246	± 48.264759	$\pm 121.24245i$	± 35.40873
0.06	0.509117	0.2235341	± 49.266985	$\pm 122.1156i$	$\pm 35.50873i$

Table 4: The effect of triaxiality on the location and stability of out-of-plane equilibrium points of Kruger 60 for $e = 0.4100, a = 0.5894, \mu = 0.3937$ and $q_2 = 0.9996$

Triaxiality				Out-of-plane points		Roots of the characteristic equation		
σ_1	σ_2	σ_3	σ_4	ξ	$\pm\zeta$	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
0.0	0.002	0.002	0.001	0.94663	0.22088	$-37.25764 \pm 21.3724i$	$0 \pm 41.8365i$	$37.25764 \pm 21.3724i$
2				5	7			

0.0 3	0.025	0.003	0.002	0.94723 3	0.22066 3	-38.6453±22.19753	0±43.2243	38.6453±-22.19753
0.0 4	0.035	0.004	0.003	0.94783 4	0.22024 7	-51.478±28.1867i	0±51.3443	51.478±28.1867i
0.0 5	0.045	0.005	0.004	0.95075 5	0.21996 5	-100.255±55.1194i	0±113.603	100.255±55.1194i
0.0 6	0.055	0.006	0.005	0.95531 4	0.21875 4	-134.56±62.55370i	0±154.122	134.56±62.55370i

Table 5: The effect of belt (M_b) on the location and stability of out-of-plane equilibrium points of Kruger-60 for $e = 0.4100, a = 0.5894, \mu = 0.3937, q_2 = 0.9996, \sigma_1=0.02, \sigma_2=0.015, \sigma_3=0.003, \sigma_4=0.002$

M_b	Out-of-Plane points		Roots of the characteristic equation		
	ξ	$\pm\zeta$	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
0.001	0.223211	0.194723	± 46.473223	± 69.2802	$\pm 56.4778i$
0.002	0.223200	0.194841	± 46.55631	± 75.5432	$\pm 62.3342i$
0.003	0.223230	0.194981	± 46.50467	± 76.775	$\pm 71.6184i$
0.004	0.223245	0.195274	± 46.36982	± 76.4643	± 89.6221
0.005	0.223297	0.195613	± 46.27746	± 111.675	± 94.3398

Table 6: The Combined effect of the perturbations on the location and stability of out-of-plane equilibrium points for $e=0.3, a=0.34$

(a)

Triaxiality				Radiation Factor	Belt	Mass ratio
σ_1	σ_2	σ_3	σ_4	q_2	M_b	μ
0.002	0.001	0.0002	0.0001	0.9972	0.001	0.0375
0.003	0.002	0.0003	0.0002	0.9976	0.002	0.0380
0.004	0.003	0.0004	0.0003	0.9980	0.003	0.0385
0.005	0.004	0.0005	0.0004	0.9984	0.004	0.0390
0.006	0.005	0.0006	0.0005	0.9988	0.005	0.0395

(b)

out-of-plane points		The characteristic Roots		
ξ	$\pm\zeta$	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
0.567412	0.185532	± 197.963	± 17.4552	$\pm 77.3282i$
0.567395	0.193443	± 211.495	± 70.5671	$\pm 83.8671i$
0.567382	0.206738	± 258.757	± 70.4693	$\pm 83.7942i$
0.567374	0.215372	± 567.09	± 71.3939	$\pm 80.133i$
0.567367	0.220541	± 667.430	± 101.4056	$\pm 22.4906i$

The motion of a third body under the influence of triaxiality and a radiating smaller primary together with a circumbinary disc has been described in equation (1)-(4). The positions of out-of-plane equilibrium points are given in equation (14) and (15) and are obtained numerically by approximating the series expansion in the triaxiality coefficient to third order term with the aid of the software MATHEMATICA 10.4. The stability of these points is determined numerically using equation (16). Table 1 shows the properties of the binary system (Kruger-60 and Xi-bootis) used in demonstrating the dynamics of the particle under investigation. The effect of triaxiality on the location and stability of out-of-plane equilibrium points of Xi-bootis is shown in Table 2 and Figures 1 & 2. Due to increase of triaxiality the infinitesimal mass moves away from the barycenter along the ξ -axis while along ζ -axis it moves towards the origin this shows that the particle is unstable, which is shown by the characteristic roots of the system which are complex or real roots. The equilibrium points are stable only if the six roots λ_i ($i=1,2,3,4,5,6$) are purely imaginary roots or complex roots with negative real parts and are unstable if λ_i ($i=1,2,3,4,5,6$) are complex or real roots (Szehebely,1967).

Table 3 shows the points $L_{6,7}$ shift towards the axis (origin), with an increase in gravitational potential from the belt of Xi-Bootis. An increase in the triaxiality factor of Kruger-60 (Table 4) moves $L_{6,7}$ away from the origin, while in the case of belt (as shown in Table 5) the point moves towards the origin. The combined effect of all the parameters are shown in Table 6. Table 6a contains the arbitrary values for the parameters and Table 6b shows their effects on out-of- plane points and their stability, As can be seen in Table 6a the out-of- plane equilibrium points shifts towards the ζ -axis with increasing values of all parameters together and they are unstable. The effects of triaxiality and the belt on the two above systems can be observed in Table 2-6. These Tables shows that with increasing values of triaxiality and belt the out-of-plane points (in the both system) move towards the ζ axis and they are unstable since the six roots λ_i ($i=1,2,3,4,5,6$) are not purely imaginary roots or complex roots with negative real parts. This instability has been confirmed by (Douskos & Markellos (2006); Kushvah (2008); Singh & Umar (2013))

CONCLUSION

We have established the existence of out of plane equilibrium points and their stability in the framework of (ER3BP) when the primaries are triaxial, with only the smaller primary radiating and surrounded by a belt. It is found that the positions are affected by triaxiality, radiation and the belt. The out-of-plane equilibrium points are unstable.

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