Journal of Basics and Applied Sciences Research (JOBASR)
ISSN (print): 3026-9091, ISSN (online): 1597-9962
Volume 4(2) March 2026
DOI: https://dx.doi.org/10.4314/jobasr.v4i2.30

Ratio-Product cum Regression Variance Estimators in the Presence of Random Nonresponse

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ABSTRACT
The variability of a population under study can be estimated using variance estimators so that better policies can be devised. This study aimed to propose a new approach for estimating the variance of a finite population using auxiliary information, while accounting for two common scenarios of random non-response in simple random sampling is suggested. The proposed estimator was derived using linear combination approach. The power series and exponential series were used in deriving the properties of the estimator (bias and mean square error). The properties were compared theoretically with existing estimators, up to the first level of approximation. The theoretical conditions for efficiency under the two scenarios of random non-response were established. The criteria of mean square error and percentage relative efficiency were used in assessing the performance of the estimator. The empirical findings using real datasets revealed that the proposed estimators performed better than the existing variance estimators with minimum mean square error and higher percentage relative efficiency values. Thus, the proposed estimator can be utilized to estimate population variability in real-world scenarios where random non-response occurs.

Keywords:
Non-response;
Auxiliary variable;
Efficiency;
Bias;
Mean square error.

INTRODUCTION

Estimating population variance is crucial in the real-world and has received considerable attention. Understanding variation is vital for nations to create economic policies, plan agriculture, and maintains quality control in industries like manufacturing and pharmaceuticals (Muhammad & Oyeyemi, 2025a; Muhammad et al., 2022).

Supplementary information boosts survey estimates accuracy and efficiency across different sampling methods. Recent research has developed ways to leverage auxiliary variables, like using extreme values and ranks, to improve survey sampling (Muhammad *et al*., 2021; Zakari *et al*., 2020; Muhammad *et al*., 2022; Zakari & Muhammad, 2022; Zakari *et al*., 2023; Muhammad *et al*., 2023; Zakari & Muhammad, 2023; Oyeyemi *et al*., 2023; Audu *et al*., 2023; Muhammad, 2023; Muhammad *et al*., 2025; Muhammad & Oyeyemi, 2025b).

It is often challenging especially during human surveys to collect data from every single unit in the sample, leading to incomplete data due to non-response. For instance, the chosen unit might not be available on the first try, or may be unreachable even if contacted, resulting in incomplete data. Relying only on accessible responses is often inadequate because many respondents choose not to cooperate. Non-response is the term for the consequent incompleteness, which can occasionally be so significant that it may lower the estimates' accuracy. Singh *et al.* (2012), Singh and Joarder (1998), Bandyopadhyay and Singh (2015), Chaudhary *et al.* (2022), among others, have made contributions in this area regarding non-response errors.

However, the existing estimators suggested by Singh and Joarder (1998), Singh *et al.* (2012), cannot provide accurate estimates when there is negative correlation between the study and auxiliary variables, while the estimators suggested by Bandyopadhyay and Singh (2015), and Chaudhary *et al.* (2022) cannot provide accurate estimates when there is positive correlation. In addition, the errors in these estimators are still extreme which may potentially lead to inaccurate results. To address these,

How to cite this article: Isah M., Mannir A., Adefemi E. & Mujtaba S. (2026). Ratio-Product cum Regression Variance Estimators in the Presence of Random Nonresponse. *Journal of Basics and Applied Sciences Research*, 4(2), 307-319. https://dx.doi.org/10.4314/jobasr.v4i2.30

generalized variance estimator under simple random sampling is proposed in this study. The proposed estimator in this study possessed minimum mean square error and higher percentage relative efficiency values and thus provides more flexible and efficient estimates.

Existing estimators were developed by many scholars using different schemes and strategies, including a conventional unbiased variance estimator as

$$T_0^* = \frac{1}{n-r-1}\sum_{i=1}^{n-r}(y_i - \bar{y}^*)^2 \tag{2}$$

The estimator's (T_0^{**}) variance is given as

$$Var(T_0^*) \cong S_y^4\theta^*\left(\lambda_{40}-1\right) \tag{3}$$

The usual ratio population variance estimator revised by Singh and Joarder (1998) is expressed as:

$$T_1^{**} = s_y^{*2}\left(\frac{S_x^2}{s_x^{*2}}\right) \tag{4}$$

The bias and MSE of the usual ratio estimator T_1^{**}, up to first degree of approximation are, respectively, given as:

$$Bias\left(T_1^{**}\right) \cong S_y^2\theta^*\left[\left(\lambda_{04}-1\right)-\left(\lambda_{22}-1\right)\right] \tag{5}$$

$$MSE\left(T_1^{**}\right) \cong S_y^4\theta^*\left[\left(\lambda_{40}-1\right)+\left(\lambda_{04}-1\right)-2\left(\lambda_{22}-1\right)\right] \tag{6}$$

The usual product population variance estimator defined by Murthy (1964) is expressed as:

$$T_2^{**} = s_y^{*2}\left(\frac{s_x^{*2}}{S_x^2}\right) \tag{7}$$

The expression for the bias and MSE of the estimator T_2^{**} are given as:

$$Bias\left(T_2^{**}\right) \cong \theta^*\left(\lambda_{22}-1\right) \tag{8}$$

$$MSE\left(T_2^{**}\right) \cong S_y^4\theta^*\left[\left(\lambda_{40}-1\right)+\left(\lambda_{04}-1\right)+2\left(\lambda_{22}-1\right)\right] \tag{9}$$

The usual regression variance estimator defined by Isaki (1983) is given as:

$$T_3^{**} = s_y^{*2} + b_{(s_y^2,s_x^2)}\left(S_x^2 - s_x^{*2}\right) \tag{10}$$

where $b_{(s_y^2,s_x^2)}$ connote the regression coefficient.

The estimator's (T_3^{**}) MSE is given as:

$$MSE\left(T_3^{**}\right) \cong S_y^4\theta^*\left(\lambda_{40}-1\right)\left[1-\rho^2_{(s_y^2,s_x^2)}\right] \tag{11}$$

where $\rho_{(s_y^2,s_x^2)} = \left(\lambda_{22}-1\right)/\sqrt{\left(\lambda_{40}-1\right)\left(\lambda_{04}-1\right)}$ is the correlation coefficient.

Singh and Joarder (1998) suggested another population variance estimator as:

$$T_4^{**} = s_y^{*2}\frac{S_x^2}{s_x^2} + \alpha\left(\frac{s_x^{*2}}{S_x^2}-1\right) \tag{12}$$

where α is a suitably chosen constants such that the mean square error of T_4^{**} is minimum. The expression for the bias, the optimum value of α and the estimator's (T_4^{**}) MSE are respectively given as:

$$Bias\left(T_4^{**}\right) \cong S_y^2\theta^*\left[\left(\lambda_{04}-1\right)-\left(\lambda_{22}-1\right)\right] \tag{13}$$

$$\alpha = \frac{\left(\frac{1}{nq+2p}-\frac{1}{n}\right)S_y^2\left(\lambda_{22}-1\right)}{\theta^*\left(\lambda_{04}-1\right)} \tag{14}$$

$$MSE\left(T_4^{**}\right)_{\min} \cong MSE\left(T_1^{**}\right) - \frac{\left(\frac{1}{nq+2p}-\frac{1}{n}\right)^2 S_y^4\left(\lambda_{22}-1\right)^2}{\theta^*\left(\lambda_{04}-1\right)} \tag{15}$$

Chaudhary *et al.*, (2022) suggested a family of factor type estimators for estimating the population variance as:

$$T_\alpha^{**} = s_y^{*2}\left[\frac{\left(A+C\right)S_x^2 + fBs_x^{*2}}{\left(A+fB\right)S_x^2 + Cs_x^{*2}}\right] \tag{16}$$

where:

$$A = \left(\alpha-1\right)\left(\alpha-2\right),$$

$$B = \left(\alpha-1\right)\left(\alpha-4\right), C = \left(\alpha-2\right)\left(\alpha-3\right)\left(\alpha-4\right)$$

for $\alpha > 0$ and $f = \frac{n}{N}$.

The estimator's bias and minimum MSE are respectively given as:

$$Bias(T_\alpha^{**}) \cong S_y^2\theta^*\left[\left(\lambda_{04}-1\right)\phi_1(\alpha)\phi_2(\alpha)-\left(\lambda_{22}-1\right)\phi(\alpha)\right] \tag{17}$$

$$MSE(T_\alpha^{**})_{\min} \cong S_y^4\theta^*\left[\left(\lambda_{40}-1\right)-\frac{\left(\lambda_{22}-1\right)^2}{\left(\lambda_{04}-1\right)}\right] \tag{18}$$

where:

$$\phi_1(\alpha) = \frac{fB}{\left(A+fB+C\right)}, \qquad \phi_2(\alpha) = \frac{C}{\left(A+fB+C\right)}$$

and $\phi(\alpha) = \frac{\left(\lambda_{22}-1\right)}{\left(\lambda_{04}-1\right)}$

MATERIALS AND METHODS

Simple Random Sampling Scheme

The following scheme was adopted in suggesting the new methodology; consider a population $U = \{U_1, , , , , U_N\}$ consisting of N units, and a sample of size n is selected without replacement. Supposed (y_i, x_i) denote the observation of the variable of interest Y and the auxiliary character X. The population means of Y and X are respectively represented by Let $\overline{Y}$ and $\overline{X}$. It was assumed that the auxiliary character's population mean and variance ($\overline{X}$ and S_x^2) are known. Let the variable of interest's population variance be represented by S_y^2.

Let $s_y^{*2} = \sum_{i=1}^{n-r}(y_i - \bar{y}^*)^2/(n-r-1)$ and $s_x^{*2} = \sum_{i=1}^{n-r}(x_i - \bar{x}^*)^2/(n-r-1)$ be respectively the study and auxiliary variables' sample mean square (SME) on the basis of the responding part of sample, $S_y^2 = \sum_{i=1}^{N}\left(y_i - \overline{Y}\right)^2/(N-1)$ and $S_x^2 = \sum_{i=1}^{N}\left(x_i - \overline{X}\right)^2/(N-1)$ be respectively the study and auxiliary variables' population mean squares. The correlation coefficient between the variable of interest and auxiliary character is donated by ρ_{yx}. The population coefficients of variation of Y and X is donated by $C_y = S_y/\overline{Y}$ and $C_x = S_x/\overline{X}$ be, where the coefficient of covariance between Y and X is donated by $C_{yx} = S_{yx}/\overline{Y}\,\overline{X}$.

Random Non-response

The sampling condition known as random non-response was extensively discussed by Singh & Joarder (1998). Supposed $r\{r = 0,1,\ldots,(n-2)\}$ represent the number units of size *n* in a random sample on which information could not be accessed and hence $(n-r)$ are the remaining units drawn from the population of size *N* by SRSWOR scheme. In order to find the unbiased estimate of the population variance, it was assumed that $r < (n-1)$. Further, let *p denote* the non-response probability from $(n-2)$ non-response cases. Thus, the *r's* probability distribution is given as

$$P(r) = \frac{(n-r)}{(nq+2p)}\,{}^{n-2}C_r\,p^r q^{n-2-r};\; r = 0,1,2,\ldots,(n-2) \qquad (19)$$

where $q = 1-p$ and ${}^{n-2}C_r$ represents the r's number of ways from total possible $(n-2)$ responses. We define the sampling fraction due to complete response and random non-response, respectively, as: $\theta = \left(\frac{1}{n} - \frac{1}{N}\right)$ and $\theta^* = \left(\frac{1}{nq+2p} - \frac{1}{N}\right)$.

Proposed Estimator

Case I

In this section, when there is random non-response on both the variable of interest and auxiliary character, a mixture of product, ratio and exponential-type estimators were combined in proposing the new generalized estimator was proposed as:

$$G_{prop(i)}^{**} = \left\{\omega_{1i}s_y^{*2}\left[\frac{1}{2}\left(\frac{S_x^2}{s_x^{*2}} + \frac{s_x^{*2}}{S_x^2}\right)\right]^{\phi} + \omega_{2i}\left(S_x^2 - s_x^{*2}\right)\right\}\exp\left\{\frac{S_x^2 - s_x^{*2}}{S_x^2 + s_x^{*2}}\right\} \qquad (20)$$

where $\omega_{ji}\ (i,j = 1,2)$ are unknown constants whose values to be determined, and ϕ is suitably chosen constant. The two values of ϕ $(\phi = 1\ and\ 2)$ were chosen using power modification strategy to assess the performance of the estimator under two different scenarios.

Properties of the Proposed Estimator

In order to find the bias and MSE of the estimator, we consider the following relative error terms:

$$s_y^{*2} = S_y^2\left(1+e_0^*\right),\ s_x^{*2} = S_x^2\left(1+e_1^*\right)$$

such that

$E\left(e_0^*\right) = E\left(e_1^*\right) = 0$, $E\left(e_0^{*2}\right) = \theta^*\left(\lambda_{40}-1\right)$, $E\left(e_1^{*2}\right) = \theta^*\left(\lambda_{04}-1\right)$, $E\left(e_0^*e_1^*\right) = \theta^*\left(\lambda_{22}-1\right)$ and

$$\theta^* = \left(\frac{1}{nq+2p} - \frac{1}{N}\right)$$

Expressing the $G_{prop(i)}^{**}$ in terms of e_i^* $(i = 0,\ 1)$ we can write (20) as

$$G_{prop(i)}^{**} = \left\{\begin{array}{l}\omega_{1i}S_y^2\left(1+e_0^*\right)\dfrac{1}{2^{\phi}}\left[\begin{array}{l}\left(1+e_1^*\right)^{-1}\\+1+e_1^*\end{array}\right]^{\phi}\\-\omega_{2i}S_x^2e_1^*\end{array}\right\}\exp\left\{\frac{-S_x^2e_1^*}{2S_x^2+S_x^2e_1^*}\right\} \qquad (21)$$

By some appropriate simplifications, we get:

$$\left(G_{prop}^{**} - S_y^2\right) = \left\{\begin{array}{l}\omega_{1i}S_y^2 + \omega_{1i}S_y^2e_0^* + \dfrac{\phi}{2}\omega_{1i}S_y^2e_1^{*2}\\-\omega_{2i}S_x^2e_1^*\end{array}\right\}\left[1 - \frac{e_1^*}{2} + \frac{3e_1^{*2}}{8}\right] \qquad (22)$$

Multiplying out the RHS of (22) and approximating, we have

$$\left(G_{prop(i)}^{**} - S_y^2\right) = \left(\omega_{1i}-1\right)S_y^2 + \omega_{1i}S_y^2\left(\begin{array}{l}e_0^* - \dfrac{e_1^*}{2}\\-\dfrac{e_0^*e_1^*}{2} + \left[\dfrac{\phi}{2}+\dfrac{3}{8}\right]e_1^{*2}\end{array}\right) - \omega_{2i}S_x^2\left(e_1^* - \frac{e_1^{*2}}{2}\right) \qquad (23)$$

where $\eta = \left[8^{-1}(4\phi+3)\right]$

Taking expectation of (23), the estimator's bias is given as:

$$Bias\left(G^{**}_{prop(i)}\right) = (\omega_{1i}-1)S_y^2 + \theta^*\left\{\begin{array}{l}\omega_{1i}S_y^2\left[\begin{array}{l}\eta(\lambda_{04}-1)\\ -\frac{1}{2}(\lambda_{22}-1)\end{array}\right]\\ +\frac{1}{2}\omega_{2i}S_x^2(\lambda_{04}-1)\end{array}\right\}$$

(24)

By squaring and taking expectation of (24), the estimator's MSE is given as:

$$MSE\left(G^{**}_{prop(i)}\right) = \left\{\begin{array}{l}(\omega_{1i}-1)^2 S_y^4 + \omega_{1i}^2 S_y^4\theta^*\\ \left[(\lambda_{40}-1)-2(\lambda_{22}-1)+4^{-1}(8\eta+1)(\lambda_{04}-1)\right]\\ +\omega_{2i}^2 S_x^4\theta^*(\lambda_{04}-1)-\omega_{1i}S_y^4\theta^*\\ \left[2\eta(\lambda_{04}-1)-(\lambda_{22}-1)\right]\\ -2\omega_{1i}\omega_{2i}S_y^2S_x^2\theta^*\\ \left[(\lambda_{22}-1)-(\lambda_{04}-1)\right]-\omega_{2i}S_y^2S_x^2\theta^*(\lambda_{04}-1)\end{array}\right\} \quad (25)$$

Minimizing (25), the optimum values of ω_{1i} and ω_{2i} are obtained as:

$$\omega^{**}_{1i(opt)} = \frac{1+\left[2^{-1}(2\eta-1)\theta^*(\lambda_{04}-1)\right]}{1+\left[4^{-1}(8\eta-3)\theta^*(\lambda_{04}-1)\right] +\theta^*\left[(\lambda_{40}-1)-\frac{(\lambda_{22}-1)^2}{(\lambda_{04}-1)}\right]}$$

and

$$\omega^{**}_{2i(opt)} = \frac{S_y^2}{S_x^2}\left\{\frac{1}{2}-\omega^{**}_{1i(opt)}\left(1-\frac{(\lambda_{22}-1)}{(\lambda_{04}-1)}\right)\right\}$$

Substituting the optimum values of ω_{1i} and ω_{2i} into (25), the estimator's minimum *MSE* is obtained as:

$$MSE\left(G^{**}_{prop(i)}\right)_{\min} = S_y^4\left\{\left(1-\frac{\theta^*(\lambda_{04}-1)}{4}\right)-\frac{\left[1+2^{-1}(2\eta-1)\theta^*(\lambda_{04}-1)\right]^2}{\left(\begin{array}{l}1+\left(2\eta-\frac{3}{4}\right)\theta^*(\lambda_{04}-1)+\\ \theta^*\left[(\lambda_{40}-1)-\frac{(\lambda_{22}-1)^2}{(\lambda_{04}-1)}\right]\end{array}\right)}\right\} \quad (26)$$

Special Cases:

For $i=1, then\ \phi=1$. Therefore, the proposed estimator in (20) becomes

$$G^{**}_{prop(1)} = \left\{\begin{array}{l}\omega_{11}s_y^{*2}\left[\frac{1}{2}\left(\frac{S_x^2}{s_x^{*2}}+\frac{s_x^{*2}}{S_x^2}\right)\right]+\\ \omega_{21}\left(S_x^2-s_x^{*2}\right)\end{array}\right\}\exp\left\{\frac{S_x^2-s_x^{*2}}{S_x^2+s_x^{*2}}\right\} \quad (27)$$

The optimum values of ω_{11} and ω_{21} for estimator in (27), are, respectively, given as:

$$\omega^{**}_{11(opt)} = \frac{1+\frac{3}{8}\theta^*(\lambda_{04}-1)}{1+\theta^*(\lambda_{04}-1)+\theta^*\left((\lambda_{40}-1)-\frac{(\lambda_{22}-1)^2}{(\lambda_{04}-1)}\right)}$$

and

$$\omega^{**}_{21(opt)} = \frac{S_y^2}{S_x^2}\left\{\frac{1}{2}-\omega^{**}_{11(opt)}\left(1-\frac{(\lambda_{22}-1)}{(\lambda_{04}-1)}\right)\right\}$$

The minimum MSE is given as:

$$MSE\left(G^{**}_{prop(1)}\right)_{\min} = S_y^4\left\{\left(1-\frac{\theta^*(\lambda_{04}-1)}{4}\right)-\frac{\left[1+\frac{3}{8}\theta^*(\lambda_{04}-1)\right]^2}{\left[1+\theta^*(\lambda_{04}-1)+\theta^*\left((\lambda_{40}-1)-\frac{(\lambda_{22}-1)^2}{(\lambda_{04}-1)}\right)\right]}\right\} \quad (28)$$

For $i=2$, then $\phi=2$. Therefore the proposed estimator in (20) becomes

$$G^{**}_{prop(2)} = \left\{\begin{array}{l}\omega_{12}s_y^{*2}\left[\frac{1}{2}\left(\frac{S_x^2}{s_x^{*2}}+\frac{s_x^{*2}}{S_x^2}\right)\right]^2+\\ \omega_{22}\left(S_x^2-s_x^{*2}\right)\end{array}\right\}\exp\left\{\frac{S_x^2-s_x^{*2}}{S_x^2+s_x^{*2}}\right\} \quad (29)$$

The optimum values of ω_{12} and ω_{22} that minimizes the estimator in (29), are respectively given as:

$$\omega^{**}_{12(opt)} = \frac{1+\frac{7}{8}\theta^*(\lambda_{04}-1)}{1+2\theta^*(\lambda_{04}-1)+\theta^*\left((\lambda_{40}-1)-\frac{(\lambda_{22}-1)^2}{(\lambda_{04}-1)}\right)}$$

and

$$\omega^{**}_{22(opt)} = \frac{S_y^2}{S_x^2}\left\{\frac{1}{2}-\omega^{**}_{12(opt)}\left(1-\frac{(\lambda_{22}-1)}{(\lambda_{04}-1)}\right)\right\}$$

The minimum MSE is given as:

$$MSE\left(G_{prop(2)}^{**}\right)_{\min} = S_y^4\left\{\left(1-\frac{\theta^*(\lambda_{04}-1)}{4}\right)-\frac{\left[1+\frac{7}{8}\theta^*(\lambda_{04}-1)\right]^2}{\left[1+2\theta^*(\lambda_{04}-1)+\theta^*\left((\lambda_{40}-1)-\frac{(\lambda_{22}-1)^2}{(\lambda_{04}-1)}\right)\right]}\right\} \quad (30)$$

Case II

In this section, the ratio-product cum regression estimator of population variance is studied when the random non-response affects the variable of interest only. The proposed estimator is given as:

$$G_{prop(i)}^* = \left\{\begin{array}{l}\omega_{1i}s_y^{*2}\left[\frac{1}{2}\left(\frac{S_x^2}{s_x^2}+\frac{s_x^2}{S_x^2}\right)\right]^{\phi}+\\ \omega_{2i}\left(S_x^2-s_x^2\right)\end{array}\right\}\exp\left\{\frac{S_x^2-s_x^2}{S_x^2+s_x^2}\right\} \quad (31)$$

where ω_{ji} $(i,j=1,2)$ are unknown parameters to be determined and ϕ is a constant suitably chosen. The two values of ϕ $(\phi = 1\ and\ 2)$ were chosen using power modification strategy to assess the performance of the estimator under two different scenarios.

Properties of the Proposed Estimator

To obtain the estimator's bias and *MSE*, the following relative error terms are considered:

$$s_y^{*2} = S_y^2\left(1+e_0^*\right),\ s_x^2 = S_x^2\left(1+e_1\right)$$

such that

$$E\left(e_0^*\right)=E\left(e_1\right)=0, \qquad E\left(e_0^{*2}\right)=\theta^*\left(\lambda_{40}-1\right),$$

$$E\left(e_1^2\right)=\theta\left(\lambda_{04}-1\right) \qquad E\left(e_0^*e_1\right)=\theta\left(\lambda_{22}-1\right),$$

$$\theta=\left(\frac{1}{n}-\frac{1}{N}\right) \text{ and } \theta^*=\left(\frac{1}{nq+2p}-\frac{1}{N}\right)$$

Expressing the estimator $G_{prop(i)}^*$ in terms of e_i^* $(i = 0,\ 1)$ we can write (31) as

$$G_{prop(i)}^* = \left\{\begin{array}{l}\omega_{1i}S_y^2\left(1+e_0^*\right)\frac{1}{2^{\phi}}\\ \left[\left(1+e_1\right)^{-1}+1+e_1\right]^{\phi}-\omega_{2i}S_x^2e_1\end{array}\right\}\exp\left\{\frac{-S_x^2e_1}{2S_x^2+S_x^2e_1}\right\} \quad (32)$$

By some appropriate simplifications, we get:

$$\left(G_{prop}^*-S_y^2\right) = \left\{\begin{array}{l}\omega_{1i}S_y^2+\omega_{1i}S_y^2e_0^*+\\ \frac{\phi}{2}\omega_{1i}S_y^2e_1^2-\omega_{2i}S_x^2e_1\end{array}\right\}\left[1-\frac{e_1}{2}+\frac{3e_1^2}{8}\right] \quad (33)$$

Expanding the RHS of (33), we have

$$\left(G_{prop(i)}^*-S_y^2\right) = \left(\omega_{1i}-1\right)S_y^2+\omega_{1i}S_y^2\left(\begin{array}{l}e_0^*-\frac{e_1}{2}\\ -\frac{e_0^*e_1}{2}+\left[\frac{\phi}{2}+\frac{3}{8}\right]e_1^2\end{array}\right)- \omega_{2i}S_x^2\left(e_1-\frac{e_1^2}{2}\right) \quad (34)$$

where $\eta=\left[8^{-1}(4\phi+3)\right]$

Taking expectation of (34), the estimator's bias is obtained as:

$$Bias\left(G_{prop(i)}^*\right) = \left(\omega_1-1\right)S_y^2+\theta\left\{\begin{array}{l}\omega_{1i}S_y^2\left[\begin{array}{l}\eta\left(\lambda_{04}-1\right)\\ -\frac{1}{2}\left(\lambda_{22}-1\right)\end{array}\right]+\\ \frac{1}{2}\omega_{2i}S_x^2\left(\lambda_{04}-1\right)\end{array}\right\} \quad (35)$$

Squaring, taking expectation and simplify (50), the MSE of the estimator is obtained as:

$$MSE\left(G_{prop(i)}^*\right) = \left\{\begin{array}{l}\left(\omega_{1i}-1\right)^2S_y^4+\omega_{1i}^2S_y^4\\ \left[\theta^*\left(\lambda_{40}-1\right)-2\theta\left(\lambda_{22}-1\right)+4^{-1}(8\eta+1)\theta\left(\lambda_{04}-1\right)\right]\\ +\omega_{2i}^2S_x^4\theta\left(\lambda_{04}-1\right)-\omega_{1i}S_y^4f\left[2\eta\left(\lambda_{04}-1\right)-\left(\lambda_{22}-1\right)\right]\\ -2\omega_{1i}\omega_{2i}S_y^2S_x^2\theta\left[\left(\lambda_{22}-1\right)-\left(\lambda_{04}-1\right)\right]-\omega_{2i}S_y^2S_x^2\theta\left(\lambda_{04}-1\right)\end{array}\right\} \quad (36)$$

Minimizing (36), the optimum values of ω_{1i} and ω_{2i} are obtained as:

$$\omega_{1i(opt)}^* = \frac{1+\left[2^{-1}\left(2\eta-1\right)\theta\left(\lambda_{04}-1\right)\right]}{1+\left[4^{-1}\left(8\eta-3\right)\theta\left(\lambda_{04}-1\right)\right]+\left[\theta^*\left(\lambda_{40}-1\right)-\frac{\theta\left(\lambda_{22}-1\right)^2}{\left(\lambda_{04}-1\right)}\right]}$$

and

$$\omega_{2i(opt)}^* = \frac{S_y^2}{S_x^2}\left\{\frac{1}{2}-\omega_{1i(opt)}^*\left(1-\frac{\left(\lambda_{22}-1\right)}{\left(\lambda_{04}-1\right)}\right)\right\}$$

Putting ω_1 and ω_2 into (36), the estimator's minimum *MSE* is obtained as:

$$MSE\left(G^*_{prop(i)}\right)_{\min} = S_y^4\left\{\left(1-\frac{\theta(\lambda_{04}-1)}{4}\right) - \frac{\left[1+2^{-1}(2\eta-1)\theta(\lambda_{04}-1)\right]^2}{\left(1+\left(2\eta-\frac{3}{4}\right)\theta(\lambda_{04}-1)+\left[\theta^*(\lambda_{40}-1)-\frac{\theta(\lambda_{22}-1)^2}{(\lambda_{04}-1)}\right]\right)}\right\} \quad (37)$$

Special Cases:
For $i = 1$, then $\phi = 1$. Therefore the proposed estimator in (31) becomes

$$G^*_{prop(1)} = \left\{\begin{array}{l}\omega_{11}s_y^{*2}\left[\frac{1}{2}\left(\frac{S_x^2}{s_x^2}+\frac{s_x^2}{S_x^2}\right)\right] \\ +\omega_{21}\left(S_x^2-s_x^2\right)\end{array}\right\}\exp\left\{\frac{S_x^2-s_x^2}{S_x^2+s_x^2}\right\} \quad (38)$$

The expression for the ω_{11} and ω_{21} are respectively given as:

$$\omega^*_{11(opt)} = \frac{1+\frac{3}{8}\theta(\lambda_{04}-1)}{1+\theta(\lambda_{04}-1)+\left(\theta^*(\lambda_{40}-1)-\frac{\theta(\lambda_{22}-1)^2}{(\lambda_{04}-1)}\right)}$$

and

$$\omega^*_{21(opt)} = \frac{S_y^2}{S_x^2}\left\{\frac{1}{2}-\omega^*_{11(opt)}\left(1-\frac{(\lambda_{22}-1)}{(\lambda_{04}-1)}\right)\right\}$$

The estimator's minimum MSE is given as:

$$MSE\left(G^*_{prop(1)}\right)_{\min} = S_y^4\left\{\left(1-\frac{\theta(\lambda_{04}-1)}{4}\right) - \frac{\left[1+\frac{3}{8}\theta(\lambda_{04}-1)\right]^2}{\left[1+\theta(\lambda_{04}-1)+\left(\theta^*(\lambda_{40}-1)-\frac{\theta(\lambda_{22}-1)^2}{(\lambda_{04}-1)}\right)\right]}\right\} \quad (39)$$

For $i = 2$, then $\phi = 2$. Therefore the proposed estimator in equation (38) becomes

$$G^*_{prop(2)} = \left\{\begin{array}{l}\omega_{12}s_y^{*2}\left[\frac{1}{2}\left(\frac{S_x^2}{s_x^2}+\frac{s_x^2}{S_x^2}\right)\right]^2 + \\ \omega_{22}\left(S_x^2-s_x^2\right)\end{array}\right\}\exp\left\{\frac{S_x^2-s_x^2}{S_x^2+s_x^2}\right\} \quad (40)$$

The expression for the ω_{11} and ω_{21} are respectively given as:

$$\omega^*_{12(opt)} = \frac{1+\frac{7}{8}\theta(\lambda_{04}-1)}{1+2\theta(\lambda_{04}-1)+\left(\theta^*(\lambda_{40}-1)-\frac{\theta(\lambda_{22}-1)^2}{(\lambda_{04}-1)}\right)}$$

and

$$\omega^*_{22(opt)} = \frac{S_y^2}{S_x^2}\left\{\frac{1}{2}-\omega^*_{12(opt)}\left(1-\frac{(\lambda_{22}-1)}{(\lambda_{04}-1)}\right)\right\}$$

The estimator's minimum MSE is given as:

$$MSE\left(G^*_{prop(2)}\right)_{\min} = S_y^4\left\{\left(1-\frac{\theta(\lambda_{04}-1)}{4}\right) - \frac{\left[1+\frac{7}{8}\theta(\lambda_{04}-1)\right]^2}{\left[1+2\theta(\lambda_{04}-1)+\left(\theta^*(\lambda_{40}-1)-\frac{\theta(\lambda_{22}-1)^2}{(\lambda_{04}-1)}\right)\right]}\right\} \quad (41)$$

Efficiency comparisons
In this section, the theoretical comparison between the proposed estimator and some existing estimators were presented.

Case I: Random non-response on variable of interest and auxiliary character.

i. The proposed estimator's MSE is compared with usual variance estimator as:

$$Var(T_0^{**}) - MSE(G^{**}_{prop(i)})_{\min} > 0$$

$$S_y^4\theta^*(\lambda_{40}-1) - S_y^4\left\{A-\frac{B^2}{C}\right\} > 0$$

$$\left\{\theta^*(\lambda_{40}-1) - A + \frac{B^2}{C}\right\} > 0 \quad (42)$$

where:

$$A = 1-\frac{\theta^*(\lambda_{04}-1)}{4}, B = 1+\frac{(2\eta\text{-}1)}{2}\theta^*(\lambda_{04}-1) \text{ and}$$

$$C = 1+\left(2\eta-\frac{3}{4}\right)\theta^*(\lambda_{04}-1)+\theta^*\left((\lambda_{40}-1)-\frac{(\lambda_{22}-1)^2}{(\lambda_{04}-1)}\right)$$

ii. The proposed estimator's MSE is compared with usual ratio estimator as:

$$MSE(T_1^{**}) - MSE(G_{prop(i)}^{**})_{\min} > 0$$

$$S_y^4\theta^*\left[(\lambda_{40}-1)+(\lambda_{04}-1)-2(\lambda_{22}-1)\right]-S_y^4\left\{A-\frac{B^2}{C}\right\}>0$$

$$\left[\theta^*\left((\lambda_{40}-1)+(\lambda_{04}-1)-2(\lambda_{22}-1)\right)-A+\frac{B^2}{C}\right]>0 \tag{43}$$

iii. The proposed estimator's MSE is compared with usual product estimator as:

$$MSE(T_2^{**}) - MSE(G_{prop(i)}^{**})_{\min} > 0$$

$$S_y^4\theta^*\left[(\lambda_{40}-1)+(\lambda_{04}-1)+2(\lambda_{22}-1)\right]-S_y^4\left\{A-\frac{B^2}{C}\right\}>0$$

$$\left[\theta^*\left((\lambda_{40}-1)+(\lambda_{04}-1)+2(\lambda_{22}-1)\right)-A+\frac{B^2}{C}\right]>0 \tag{44}$$

iv. The proposed estimator's MSE is compared with usual regression estimator as:

$$MSE(T_3^{**}) - MSE(G_{prop(i)}^{**})_{\min} > 0$$

$$S_y^4\theta^*(\lambda_{40}-1)\left[1-\rho^2_{(s_y^2,s_x^2)}\right]-S_y^4\left\{A-\frac{B^2}{C}\right\}>0$$

$$\left\{\theta^*(\lambda_{40}-1)\left(1-\rho^2_{(s_y^2,s_x^2)}\right)-A+\frac{B^2}{C}\right\}>0 \tag{45}$$

v. The proposed estimator's MSE is compared with Singh and Joarder (1998) estimator as:

$$MSE(T_4^{**}) - MSE(G_{prop(i)}^{**})_{\min} > 0$$

$$\left\{MSE\left(T_1^{**}\right)-\frac{\left(\frac{1}{nq+2p}-\frac{1}{n}\right)^2 S_y^4(\lambda_{22}-1)^2}{\theta^*(\lambda_{04}-1)}\right\}-S_y^4\left\{A-\frac{B^2}{C}\right\}>0$$

$$\left\{MSE\left(T_1^{**}\right)-\frac{\left(\frac{1}{nq+2p}-\frac{1}{n}\right)^2 S_y^4(\lambda_{22}-1)^2}{\theta^*(\lambda_{04}-1)}-S_y^4\left\{A-\frac{B^2}{C}\right\}\right\}>0 \tag{46}$$

vi. The proposed estimator's MSE is compared with Chaudhary *et al.*, (2022) estimator as:

$$MSE(T_\alpha^{**}) - MSE(G_{prop(i)}^{**})_{\min} > 0$$

$$S_y^4\theta^*\left[(\lambda_{40}-1)-\frac{(\lambda_{22}-1)^2}{(\lambda_{04}-1)}\right]-S_y^4\left\{A-\frac{B^2}{C}\right\}>0$$

$$\left\{\theta^*\left[(\lambda_{40}-1)-\frac{(\lambda_{22}-1)^2}{(\lambda_{04}-1)}\right]-A+\frac{B^2}{C}\right\}>0 \tag{47}$$

Case II: Random non-response affects the study variable only.

i. The proposed estimator's MSE is compared with usual variance estimator as:

$$Var(T_0^{*}) - MSE(G_{prop(i)}^{*})_{\min} > 0$$

$$S_y^4\theta^*(\lambda_{40}-1)-S_y^4\left\{D-\frac{E^2}{F}\right\}>0$$

$$\left\{\theta^*(\lambda_{40}-1)-D+\frac{E^2}{F}\right\}>0 \tag{48}$$

where:

$D=1-\frac{\theta(\lambda_{04}-1)}{4}$, $E=1+\frac{(2\eta\text{-}1)}{2}\theta(\lambda_{04}-1)$ and

$$F=1+\left(2\eta-\frac{3}{4}\right)\theta(\lambda_{04}-1)+\left[\theta^*(\lambda_{40}-1)-\frac{\theta(\lambda_{22}-1)^2}{(\lambda_{04}-1)}\right]$$

ii. The proposed estimator's MSE is compared with usual ratio estimator as:

$$MSE(T_1^{*}) - MSE(G_{prop(i)}^{*})_{\min} > 0$$

$$S_y^4\left[\theta^*(\lambda_{40}-1)+\theta\left\{(\lambda_{04}-1)-2(\lambda_{22}-1)\right\}\right]-S_y^4\left\{D-\frac{E^2}{F}\right\}>0$$

$$\left[\left(\theta^*(\lambda_{40}-1)+\theta\left\{(\lambda_{04}-1)-2(\lambda_{22}-1)\right\}\right)-D+\frac{E^2}{F}\right]>0 \tag{49}$$

iii. The proposed estimator's MSE is compared with usual product estimator as:

$$MSE(T_2^{*}) - MSE(G_{prop(i)}^{*})_{\min} > 0$$

$$S_y^4\left[\theta^*(\lambda_{40}-1)+\theta\left\{(\lambda_{04}-1)+2(\lambda_{22}-1)\right\}\right]-S_y^4\left\{D-\frac{E^2}{F}\right\}>0$$

$$\left[\left(\theta^*(\lambda_{40}-1)+\theta\left\{(\lambda_{04}-1)+2(\lambda_{22}-1)\right\}\right)-D+\frac{E^2}{F}\right]>0 \tag{50}$$

iv. The proposed estimator's MSE is compared with usual regression estimator as:

$$MSE(T_3^{*}) - MSE(G_{prop(i)}^{*})_{\min} > 0$$

$$S_y^4\left[\theta^*(\lambda_{40}-1)-\frac{\theta(\lambda_{22}-1)^2}{(\lambda_{04}-1)}\right]-S_y^4\left\{D-\frac{E^2}{F}\right\}>0$$

$$\left\{\theta^*(\lambda_{40}-1)-\frac{\theta(\lambda_{22}-1)^2}{(\lambda_{04}-1)}-D+\frac{E^2}{F}\right\}>0 \tag{51}$$

v. The proposed estimator's MSE is compared with Singh and Joarder (1998) estimator as:

$$MSE(T_4^*) - MSE(G_{prop(i)}^*)_{\min} > 0$$

$$\left\{ MSE\left(T_1^*\right) - \frac{\left(\frac{1}{nq+2p} - \frac{1}{n}\right)^2 S_y^4 \left(\lambda_{22}-1\right)^2}{\theta\left(\lambda_{04}-1\right)} \right\} - S_y^4 \left\{ D - \frac{E^2}{F} \right\} > 0$$

$$\left\{ MSE\left(T_1^*\right) - \frac{\left(\frac{1}{nq+2p} - \frac{1}{n}\right)^2 S_y^4 \left(\lambda_{22}-1\right)^2}{\theta\left(\lambda_{04}-1\right)} - S_y^4 \left\{ D - \frac{E^2}{F} \right\} \right\} > 0 \qquad (52)$$

vi. The proposed estimator's MSE is compared with Chaudhary *et al.*, (2022) estimator as:

$$MSE(T_\alpha^{**}) - MSE(G_{prop(i)}^{**})_{\min} > 0$$

$$S_y^4 \left[\theta^* \left(\lambda_{40}-1\right) - \theta \frac{\left(\lambda_{22}-1\right)^2}{\left(\lambda_{04}-1\right)} \right] - S_y^4 \left\{ D - \frac{E^2}{F} \right\} > 0$$

$$\left\{ \theta^* \left(\lambda_{40}-1\right) - \theta \frac{\left(\lambda_{22}-1\right)^2}{\left(\lambda_{04}-1\right)} - D + \frac{E^2}{F} \right\} > 0 \qquad (53)$$

Since the conditions (42)–(53) are satisfied, therefore the proposed estimator performs better than all the existing estimators considered.

Data Source and Description

In this section, four real-life datasets were used to assess the performances of the proposed and existing estimators. The datasets sources, description of the variables involved (y, x, z), and the numerical values of the parameters are presented in Table 1. The computation of the mean square error and percentage relative efficiency values was analyzed using R statistical software.

Table 1: Source and Description of Data Sets

Data set	Source	Y	X
1	Shukla and Thakur (2008)	Per capita consumption of chickens in pounds in United States from 1960–1982.	Real disposable income per capita in dollars in United States from 1960–1982.
2	Sukhatme & Sukhatme (1970)	Number of literate persons in the village.	Number of workers in the village.
3	Singh and Mangat, (1996)	Leaf area for the newly developed strain of wheat.	The weight of leaves.
4	Murthy (1967)	Output for 80 factories in a region.	Fixed capital.

Table 2: Parameters of the Data Sets

Parameters	Data 1	Data 2	Data 3	Data 4
N	200	34	39	80
n	20	12	14	20
$\bar{Y}$	42.485	201.4118	26.8433	51.8264
$\bar{X}$	18.515	218.4118	106.2	11.2646
S_y^2	199.0598	23154.85561	38.9900	336.9757
S_x^2	48.5375	28123.21925	124.1286	70.6634
ρ	0.8652	0.6203457435	0.8899	0.7941
λ_{40}	2.56	3.4426	2.4032	2.2667
λ_{04}	3.74	3.18507	2.9930	2.8664
λ_{22}	2.47	2.433161	2.4882	2.2209

RESULTS AND DISCUSSION

Comparison of Estimators

The proposed and some existing estimators' empirical biases and MSEs were obtained and presented in the tables below.

Table 3: The estimators MSE and PRE values using datasets 1 and 2

		Dataset 1				Dataset 2			
		Case I		Case II		Case I		Case II	
P	Estimators	MSE	PRE	MSE	PRE	MSE	PRE	MSE	PRE
0.05	T_0	2927.30	100.00	2927.40	100.00	75360239	100.00	75360239	100.00
	T_1	2552.00	114.71	2570.68	113.87	54341933	138.68	55665308	135.38

	T_2	13585.7	21.55	13055.4	22.422	231208151	32.59	221395504	34.039
	T_3	736.01	397.72	2025.85	144.50	46359446	162.56	64234755	117.32
	T_4	2548.34	114.87	2566.82	114.04	54226963	138.97	55542613	135.68
	T_α	1447.42	202.24	1521.04	192.45	46359947.35	162.56	48185131	156.40
	$G_{prop(1)}$	977.35	299.51	1066.42	274.50	34309861	219.65	36315816	207.51
	$G_{prop(2)}$	429.09	682.22	553.882	528.51	25541790	295.05	28174605	267.48
0.10	T_0	3087.34	100.00	3087.34	100.00	80536498	100.00	80536498	100.00
	T_1	2691.53	114.71	2730.72	113.06	58074511	138.68	60841567	132.37
	T_2	14328.4	21.547	13215.4	23.36	247089115	32.594	226571763	35.51
	T_3	776.25	397.73	2185.89	141.24	49543731	162.56	69411014	116.03
	T_4	2676.23	115.36	2713.73	113.77	57604178	139.81	60305155	133.55
	T_α	1526.55	202.24	1681.08	183.65	49543399	162.56	53361390	150.93
	$G_{prop(1)}$	1007.72	306.37	1195.78	258.19	35945528	224.05	40154627	200.57
	$G_{prop(2)}$	407.80	757.07	672.476	459.10	26173535	307.70	31733528	253.79
0.15	T_0	3264.03	100.00	3264.03	100.00	86205734	100.00	86205734	100.00
	T_1	2845.57	114.71	2907.41	112.27	62162571	138.68	66510803	129.61
	T_2	15148.5	21.547	13392.1	24.373	264482551	32.594	232240999	37.12
	T_3	820.67	397.73	2362.59	138.16	53031281	162.56	75080249	114.82
	T_4	2809.53	116.18	2865.12	113.92	61077519	141.14	65186194	132.25
	T_α	1613.92	202.24	1857.78	175.70	53030927	162.56	59030626	146.04
	$G_{prop(1)}$	1038.91	314.18	1337.56	244.03	37651406	228.96	44289777	194.64
	$G_{prop(2)}$	380.25	858.40	802.548	406.71	26746030	322.31	35572806	242.34
0.20	T_0	3460.12	100.00	3460.12	100.00	92441893	100.00	92441893	100.00
	T_1	3016.51	114.71	3103.49	111.49	66659438	138.68	72746962	127.07
	T_2	16058.5	21.547	13588.2	25.464	283615332	32.594	238477159	38.763
	T_3	869.97	397.73	2558.67	135.23	56867587	162.56	81316409	113.68
	T_4	2949.26	117.32	3019.84	114.58	64676204	142.93	70150729	131.78
	T_α	1710.87	202.24	2053.86	168.47	56867206	162.56	65266785	141.64
	$G_{prop(1)}$	1070.72	323.16	1493.65	231.66	39428495	234.45	48756957	189.60
	$G_{prop(2)}$	344.89	1003.3	945.851	365.82	27239389	339.37	39726924	232.69

Table 3 presents the empirical results using data sets 1 and 2. The results obtained from dataset 1 revealed that the proposed estimators; I and II have minimum MSE and higher PRE values (977.35 and 299.51; 429.09 and 682.22) compared to the usual variance estimator (2927.30 and 100); classical variance ratio estimator (2552.00 and 114.71); classical product estimator (13585.7 and 21.55); classical regression estimator (736.01 and 397.72); Singh and Joarder (1998) ratio-type estimator (2548.34 and 114.87); and Chaudhary et al. (2022) family of factor-type variance estimator (1447.42 and 202.24). The results obtained from dataset 2 revealed that the proposed estimators; I and II have minimum MSE and higher PRE values (34309861 and 219.65; 25541790 and 295.05) compared to the usual variance estimator (75360239 and 100); classical variance ratio estimator (54341933 and 138.68); classical product estimator (231208151 and 32.59); classical regression estimator (46359446 and 162.56); Singh and Joarder (1998) ratio-type estimator (54226963 and 138.97); and Chaudhary et al. (2022) family of factor-type variance estimator (46359947.35 and 162.56). The results subsequently revealed that the proposed generalized estimators have the minimum mean square error and higher percentage relative efficiency values than some existing estimators considered under varying non-response errors (0.10, 0.15 and 0.20).

Table 4: The estimators MSE and PRE values using datasets 3 and 4

		Dataset 1				Dataset 2			
		Case I		Case II		Case I		Case II	
P	Estimators	MSE	PRE	MSE	PRE	MSE	PRE	MSE	PRE
0.05	T_0	104.495 3	100	104.50	100	5732.775	100	5732.775	100

	T_1	31.26219	334.25	36.04	289.91	3128.655	183.23	3282.593	174.6417
	T_2	474.5628	22.019	450.40	23.20	25230.64	22.73	24078.05	23.80913
	T_3	21.74316	480.59	49.37	211.66	2117.717	270.71	3047.986	188.084
	T_4	30.90943	338.07	35.67	292.98	3116.025	183.98	3269.169	175.3588
	T_α	21.74063	480.65	27.14	384.97	2118.289	270.63	2331.953	245.8358
	$G_{prop(1)}$	13.98463	747.22	19.46	537.07	1677.383	341.77	1901.71	301.4537
	$G_{prop(2)}$	2.373887	4401.86	8.75	1194.33	1099.555	521.37	1367.621	419.1785
0.10	T_0	111.9574	100	111.96	100	6105.174	100	6105.174	100
	T_1	33.49466	334.25	43.51	257.34	3331.891	183.23	3654.992	167.0366
	T_2	508.4519	22.019	457.86	24.45	26869.61	22.721	24450.45	24.96958
	T_3	23.29587	480.59	56.83	196.99	2255.283	270.71	3420.385	178.4938
	T_4	29.61716	349.31	41.85	267.51	3279.644	186.15	3595.855	169.7837
	T_α	23.29315	480.65	34.61	323.52	2255.892	270.63	2704.352	225.7537
	$G_{prop(1)}$	14.45341	774.61	25.92	431.99	1758.593	347.16	2230.513	273.7117
	$G_{prop(2)}$	1.333336	8396.89	14.76	758.33	1111.171	549.44	1678.155	363.8027
0.15	T_0	120.1535	100	120.15	100	6516.32	100	6516.32	100
	T_1	35.94672	334.25	51.70	232.40	3556.274	183.23	4066.138	160.2582
	T_2	545.6743	22.019	466.06	25.78	28679.11	22.72	24861.6	26.21038
	T_3	25.0013	480.59	65.03	184.77	2407.162	270.71	3831.531	170.0709
	T_4	32.61567	368.39	47.60	252.40	3434.376	189.74	3918.874	166.2804
	T_α	24.99838	480.65	42.80	280.72	2407.812	270.63	3115.498	209.1582
	$G_{prop(1)}$	14.89619	806.61	32.95	364.70	1844.697	353.25	2591.244	251.4746
	$G_{prop(2)}$	0.040644	295623.5	21.31	563.75	1116.876	583.44	2018.972	322.7543
0.20	T_0	129.1975	100	129.20	100	6972.591	100	6972.591	100
	T_1	38.65244	334.25	60.75	212.69	3805.283	183.23	4522.41	154.1787
	T_2	586.7473	22.02	475.10	27.19	30687.22	22.721	25317.87	27.5402
	T_3	26.88315	480.59	74.072	174.42	2575.711	270.71	4287.803	162.6146

	T_4	32.56066	396.79	52.69	245.21	3579.918	194.77	4231.084	164.7944
	T_α	26.88001	480.65	51.85	249.20	2576.407	270.63	3571.769	195.2139
	$G_{prop(1)}$	15.2992	844.47	40.63	318.02	1935.962	360.16	2988.791	233.2914
	$G_{prop(2)}$	1.5606	8278.70	28.47	453.75	1114.722	625.50	2394.73	291.164

Table 4 presented the empirical results using data sets 3 and 4. The results obtained from dataset 3 revealed that the proposed estimators; I and II have minimum MSE and higher PRE values (13.98463 and 747.22; 2.373887 and 4401.86) compared to the usual variance estimator (104.4953 and 100); classical variance ratio estimator (31.26219 and 334.25); classical product estimator (474.5628 and 22.019); classical regression estimator (21.74316 and 480.59); Singh and Joarder (1998) ratio-type estimator (30.90943 and 338.07); and Chaudhary et al. (2022) family of factor-type variance estimator (21.74063 and 480.65). The results obtained from dataset 4 revealed that the proposed estimators; I and II have minimum MSE and higher PRE values (12283505 and 338.79; 7976253 and 521.74) compared to the usual variance estimator (41615258 and 100); classical variance ratio estimator (23680147 and 175.74); classical product estimator (98423791 and 42.28); classical regression estimator (23652074 and 175.59); Singh and Joarder (1998) ratio-type estimator (23578059 and 176.50); and Chaudhary et al. (2022) family of factor-type variance estimator (23651146 and 175.95). The results subsequently revealed that the proposed generalized estimators have the minimum mean square error and higher percentage relative efficiency values than some existing estimators considered under varying non-response errors (0.10, 0.15 and 0.20).

Generalized variance estimators in single phase sampling in the presence of random non-response were developed in this study. The performance of the estimators based on MSE and PRE criteria using four real-life datasets were evaluated. The results computed using dataset 1 revealed that the new estimators; I and II have minimum MSE and higher PRE values (977.35 and 299.51; 429.09 and 682.22) compared to the usual variance estimator (2927.30 and 100); classical variance ratio estimator (2552.00 and 114.71); classical product estimator (13585.7 and 21.55); classical regression estimator (736.01 and 397.72); Singh and Joarder (1998) ratio-type estimator (2548.34 and 114.87); and Chaudhary et al. (2022) family of factor-type variance estimator (1447.42 and 202.24). The results computed using dataset 2 similarly revealed that the new estimators; I and II have minimum MSE and higher PRE values (34309861 and 219.65; 25541790 and 295.05) compared to the usual variance estimator (75360239 and 100); classical variance ratio estimator (54341933 and 138.68); classical product estimator (231208151 and 32.59); classical regression estimator (46359446 and 162.56); Singh and Joarder (1998) ratio-type estimator (54226963 and 138.97); and Chaudhary et al. (2022) family of factor-type variance estimator (46359947.35 and 162.56). Also, the results obtained from dataset 3 revealed that the new estimators; I and II have minimum MSE and higher PRE values (13.98463 and 747.22; 2.373887 and 4401.86) compared to the usual variance estimator (104.4953 and 100); classical variance ratio estimator (31.26219 and 334.25); classical product estimator (474.5628 and 22.019); classical regression estimator (21.74316 and 480.59); Singh and Joarder (1998) ratio-type estimator (30.90943 and 338.07); and Chaudhary et al. (2022) family of factor-type variance estimator (21.74063 and 480.65). Further, the results obtained from dataset 4 revealed that the new estimators; I and II have minimum MSE and higher PRE values (12283505 and 338.79; 7976253 and 521.74) compared to the usual variance estimator (41615258 and 100); classical variance ratio estimator (23680147 and 175.74); classical product estimator (98423791 and 42.28); classical regression estimator (23652074 and 175.59); Singh and Joarder (1998) ratio-type estimator (23578059 and 176.50); and Chaudhary et al. (2022) family of factor-type variance estimator (23651146 and 175.95). The results subsequently revealed that the new generalized estimators have minimum MSE and higher PRE values than existing estimators under varying non-response errors (0.05, 0.10, 0.15 and 0.20). Thus, the proposed estimators were more efficient and robust based on MSE and PRE criteria.

CONCLUSION

A generalized ratio-product cum regression-type classes of estimators of finite population variance were proposed in this study. The bias and MSE of the estimators were derived. The theoretical efficiency conditions under two realistic situations of random non-responses were also established. Empirical application showed that the new proposed estimators have least MSE and higher PRE values, and thus performed better than the existing estimators. Therefore, the proposed estimators are recommended for practical application in variance estimation under random non-response. Thus, the study contributes to the literature by developing new variance estimators that provides reliable and accurate results in estimating variations that arise in real-world scenarios under random non-response condition. Future research could extend the proposed estimators to account for measurement error and other complex survey conditions.

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