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Development of Odd Rayleigh-Frechet Distribution: Properties and Applications

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ABSTRACT
Accurate modelling of rainfall patterns is crucial for hydrological forecasting, water resource management, and climate analysis. In this study, we propose the Odd Rayleigh Fréchet Distribution (ORFD) as a novel probability model for analysing rainfall at the Piracicaba River and the average annual rainfall across Nigeria. The performance of ORFD is assessed against two competing models: Fréchet Distribution (FD) and Weibull Distribution (WD) using log-likelihood, Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC) across five months (May-September). The results show that ORFD consistently provides the best fit, exhibiting the lowest AIC and BIC values, whereas FD and WD perform less effectively and indicating numerical instability. A simulation study is conducted to evaluate the Bias and Root Mean Square Error (RMSE) of the parameter estimates for ORFD. The results demonstrate that bias and RMSE decrease as the sample size increases, confirming the consistency and efficiency of the proposed model. These findings highlight the potential of ORFD as a robust statistical tool for modelling heavy-tailed rainfall data, thereby improving hydrological predictions and decision-making processes.

Keywords:
Odd Rayleigh-Fréchet Distribution; rainfall Modelling; Heavy-tailed Distributions; Hydrological data; Parameter estimation; Model comparison; Simulation study

INTRODUCTION

Probability distributions play a crucial role in modelling real-world phenomena across diverse fields such as reliability engineering, survival analysis, environmental sciences, and financial risk assessment (Li, 2025; Qayoom *et al*., 2025). Classical distributions like the Rayleigh and Fréchet distributions have been widely used due to their ability to model skewed and heavy-tailed data (Johnson *et al*., 1994; Sadiq *et al*., 2024; Sadiq *et al*., 2023a, 2023b, 2023c, and Umar *et al*., 2026). However, real-world data often exhibit complex behaviours such as multimodality, skewness, and varying tail weights, which traditional distributions may not adequately capture (Gupta *et al*., 2010). To address such limitations, researchers have introduced generalised and extended forms of these distributions to provide better flexibility and improved fitting performance (Cordeiro& de Castro, 2011).

The Rayleigh distribution is a special case of the Weibull distribution, often used in reliability analysis and survival studies (Baaqeel *et al*., 2025; Sadiq *et al*., 2025). It is commonly applied in modelling wind speed, signal processing, and lifetime data (Papoulis 1965; Sadiq *et al*., 2024; Sadiq *et al*., 2022). However, its ability to model heavy tails is limited, which restricts its application in extreme value problems.

On the other hand, the Fréchet distribution is a well-known extreme value distribution used to model extreme events such as catastrophic financial losses, climate extremes, and material strength (Gumbel, 1958; Sadiq *et al*., 2023a, 2023b, and 2023c). It exhibits a strong right-skewed and heavy-tailed nature, making it useful for modelling maximum values in datasets. Despite its strengths, the Fréchet distribution lacks flexibility in handling datasets that require additional shape parameters for better adaptability (Hosking, 1985; Sadiq *et al*., 2023a, 2023b, and 2023c).

To improve flexibility in statistical modelling, many researchers have proposed new families of probability distributions by adding shape parameters through various transformation techniques (Shawki *et al*., 2025; Baharith & Aljuhani, 2021; Alshawarbeh, 2024). Among these, the Odd Transformation Family has gained attention as it introduces an additional parameter that improves model flexibility while retaining desirable mathematical properties (Alzagha *et al*., 2021; Sadiq *et al*., 2024). Notable extensions include the Odd Generalised Exponential family (Cordeiro& de Castro, 2011), Odd Weibull-G (Tahmasebi & Zarei, 2016), and Odd Lomax-G families (Oluyede *et al*., 2016), all of which have shown superior performance in modelling complex real-world datasets.

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Given the strengths and limitations of the Rayleigh and Fréchet distributions, there is a need to develop a new distribution that incorporates their desirable features while improving flexibility in data fitting (Sadiq *et al*., 2024; Gemeay *et al*., 2024). The Odd Rayleigh-Fréchet (ORF) distribution is proposed to address this need by combining the light-tailed nature of the Rayleigh distribution with the heavy-tailed property of the Fréchet distribution. This new distribution is expected to provide a wider range of skewness and kurtosis values to better accommodate diverse datasets, and offer improved flexibility in tail behaviour, making it suitable for applications in reliability engineering, survival analysis, finance, and environmental studies (Nadarajah & Kotz, 2006); and serve as a superior alternative to traditional Rayleigh and Fréchet models.

Park (1961) focused on deriving the moments of the generalised Rayleigh distribution (GRD), an extension of the classical Rayleigh distribution that accommodates additional flexibility in modelling. Moments, such as mean, variance, skewness, and kurtosis, are fundamental statistical measures that describe the shape and behaviour of a probability distribution (De Michele & De Bartolo, 2026; Kim & Kim, 2025). Siddiqui (1962) provided a valuable contribution to the statistical literature. Its combination of theoretical insights, parameter estimation methods, and practical applications makes it a foundational work for researchers and practitioners working with the Rayleigh distribution. Blumenson and Miller (1963) presented a foundational work that extends the Rayleigh distribution to a broader class of problems. Its combination of mathematical consistency and practical relevance makes it a valuable resource for statisticians and applied researchers alike. Beckmann (1964) made an important contribution to the statistical literature on the Rayleigh distribution and its generalisations. By exploring mathematical properties, practical applications, and theoretical advancements, the study likely provided a foundation for many subsequent works in statistical modelling and applied probability. Hoffman and Karst (1975) provided a valuable contribution to the theory and application of the Rayleigh distribution. Its emphasis on maritime applications makes it particularly relevant to the field of ship research, while its theoretical insights have broader implications for applied probability and statistics. Kuruoglu and Zerubia (2004) provided a valuable contribution to both statistical modelling and SAR image analysis. By proposing a flexible and adaptable distribution, the authors address key challenges in representing SAR image data, paving the way for more accurate and effective image processing techniques. Kundu and Raqab (2005) provided a thorough investigation of the Generalised Rayleigh Distribution and explored different methods for parameter estimation. By combining theoretical derivations with practical applications and comparative analysis, the research makes a significant contribution to statistical modelling and estimation. Rosaiah and Kantam (2005) provided a valuable contribution to the field of quality control. It combines rigorous mathematical derivations with practical applications, providing a useful tool for lifetime testing in industrial settings. Salo *et al*. (2006) investigated and derived the statistical properties of the distribution of the product of independent Rayleigh random variables (RVs). Merovci (2013) introduced the Transmuted Rayleigh Distribution (TRD**)**, an extension of the classical Rayleigh distribution. By applying the quadratic rank transmutation map (QRTM) to the Rayleigh distribution, the study creates a more flexible distribution that can model data with varying skewness and kurtosis (Khan & King, 2015). The Merovci (2013), also estimates the parameters of the TRD using the Maximum Likelihood Estimation (MLE) method and evaluates its performance through a real-life data application. Akarawak *et al*. (2013) introduced the Weibull-Rayleigh distribution, a new distribution derived by compounding the Weibull and Rayleigh distributions. This new model increases flexibility in modelling real-life phenomena where the classic Weibull or Rayleigh distributions may not provide an adequate fit (Akarawak *et al*., 2013). The authors systematically explore its mathematical properties, including its probability density function (PDF), cumulative distribution function (CDF), survival function, and hazard rate function (Akarawak *et al*., 2013). Parameter estimation is performed using the Maximum Likelihood Estimation (MLE) method. The study concludes by applying the Weibull-Rayleigh distribution to real-world data, demonstrating its superiority over existing models (Akarawak *et al*., 2013). Dey *et al*. (2014) provided a thorough exploration of the two-parameter Rayleigh distribution and compared various methods for parameter estimation. By combining theoretical derivations, simulation studies, and real-world applications, the authors make a significant contribution to the field of statistical modelling (Dey *et al*., 2014). Merovci and Elbatal (2015) provided a valuable contribution by introducing the Weibull-Rayleigh distribution. Its combination of theoretical development, parameter estimation methods, and real-world applications demonstrates its potential as a flexible and versatile tool for statistical modelling (Merovci and Elbatal, 2015). Alamatsaz *et al*. (2016) provided a comprehensive consideration of the discrete generalised Rayleigh distribution. By combining theoretical derivations with practical applications, the study makes a valuable contribution to the field of probability and statistics (Alamatsaz *et al*., 2016). The study by Alamatsaz *et al*. (2016), is a valuable resource for statisticians and researchers interested in discrete probability modelling. Ateeq *et al*. (2019) provided a valuable extension of the Rayleigh distribution and demonstrated its utility in modelling complex data. The combination of theoretical developments, parameter estimation methods, and real-world applications makes this a significant contribution to the field of statistical modelling (Ateeq *et al.*, 2019). Al-Noor and Assi (2020) make a valuable contribution to the field of statistical modelling. By introducing a flexible extension of the Rayleigh distribution and demonstrating its practical applications, the study addresses important gaps in the literature (Al-Noor & Assi, 2020). Bantan *et al.* (2020) provided a comprehensive framework for understanding and applying the unit Rayleigh distribution. By combining rigorous theoretical analysis with practical applications, the study advances both the theory and practice of statistical modelling (Bantan *et al.*, 2020). Bhat and Ahmad (2020) introduced a novel generalisation of the Rayleigh distribution, aiming to improve its flexibility and applicability to various datasets. The authors propose a new probability distribution by incorporating additional parameters into the classical Rayleigh model (Bhat & Ahmad, 2020). The authors apply the Maximum Likelihood Estimation (MLE) method for parameter estimation and evaluate the performance of the new model through applications to real-world datasets (Bhat & Ahmad 2020). Almongy *et al*. (2021) provided a valuable contribution to statistical modelling by introducing an extended Rayleigh distribution tailored to real-world applications like COVID-19 data analysis. Its combination of theoretical rigour and practical relevance makes it a significant addition to the field (Almongy *et al*., 2021). Shen*et al.* (2022) introduced a novel generalised Rayleigh distribution (GRD) and investigated its theoretical properties and applications. The GRD was developed to address the limitations of the classical Rayleigh distribution when modelling complex data structures, particularly for big data contexts (Shen*et al*., 2022). Key mathematical properties such as the probability density function (PDF), cumulative distribution function (CDF), moments, and entropy are derived (Shen*et al*., 2022). The authors employ the Maximum Likelihood Estimation (MLE) method for parameter estimation and evaluate the performance of the GRD using large-scale data from an online community (Shen*et al*., 2022). Sadiq *et al*. (2024) introduced the Odd Rayleigh-G (ORG) family of distributions, a new and flexible class of statistical models. The study explored its mathematical properties, such as the probability density function (PDF), cumulative distribution function (CDF), and moments (Sadiq *et al*., 2024). The authors applied maximum likelihood estimation (MLE**)** for parameter estimation and conducted simulation studies to evaluate its performance (Sadiq *et al*., 2024).

Dowson and Landau (1982) presented a significant contribution to multivariate analysis by introducing a method to measure the distance between multivariate normal distributions, considering both differences in their means and covariance structures. Rüschendorf (1991) delves into the theory and applications of Fréchet bounds, a cornerstone concept in probability and statistics for analysing the dependency structure of multivariate distributions. Abd-Elfattah and Omima (2009) focus on parameter estimation for the Generalised Fréchet (GF) distribution, which is widely used in modelling extreme events across various fields. Barreto-Souza *et al.* (2011) introduced the Beta Fréchet distribution (BFD**)**, which extends the Fréchet distribution by incorporating the beta generator approach. This approach allows for greater flexibility in modelling data with heavy tails, as the BFD provides an additional layer of shape parameters. Krishna *et al.* (2013) developed a significant advancement in the modelling of extreme-value data, offering both theoretical developments and practical applications. Da Silva *et al.* (2013) introduced a new lifetime probability distribution called the Gamma Extended Fréchet (GEF) distribution, developed by combining the gamma distribution with the Fréchet distribution. An in-depth exploration of key properties such as moments, quantiles, and skewness (Da Silva *et al.*, 2013). Buckley applied to enhance the flexibility of the standard Fréchet distribution in modelling real-world phenomena (Da Silva *et al*., 2013). Mahmoud and Mandouh (2013) focused on deriving the mathematical properties of the TFD, including its probability density function (PDF)**,** cumulative distribution function (CDF)**,** hazard function**,** and moment**s**. Parameter estimation is performed using the Maximum Likelihood Estimation (MLE**)** method (Mahmoud & Mandouh, 2013). Turner *et al*. (2014) make a substantial contribution to topological data analysis by defining and providing methods to compute Fréchet means for distributions of persistence diagrams (Turner *et al*., 2014), thereby enhancing the statistical tools available for studying the topology of data (Turner *et al*., 2014). Abbas and Tang (2015) provided a comprehensive and insightful analysis of the Fréchet distribution using reference priors, contributing valuable methodologies for Bayesian estimation in the context of extreme value theory (Abbas & Tang, 2015). Yousof *et al.* (2016) provided a robust distributional model, supported by thorough mathematical derivations and practical applications, thereby enriching the tools available for statistical analysis of complex data (Yousof *et al*., 2016). Mead *et al*. (2017) presented a well-structured and insightful study that significantly contributes to the field of statistical distribution theory, offering a flexible and practical model for analysing complex data across various scientific disciplines (Mead *et al*., 2017). Gorshenin and Korolev (2018) present a valuable and insightful study that advances the statistical modelling of extreme precipitation. The proposed scale mixtures of Fréchet distributions offer a flexible and effective approach for approximating the distribution of extreme precipitation events (Gorshenin & Korolev, 2018), with practical implications for risk assessment and environmental management. Yousof *et al.* (2018) present a well-structured and insightful study that significantly contributes to the field of statistical distribution theory and regression modelling, offering flexible and practical tools for analysing complex data across various scientific disciplines. Nasiru *et al*. (2019) presented a well-structured and insightful study that significantly contributes to the field of statistical distribution theory, offering a flexible and practical model for analysing extreme events and lifetime data. Abbas *et al.* (2019) offer a valuable contribution to statistical methodology in medical research by developing Bayesian estimators for the three-parameter Fréchet distribution, accommodating the complexities inherent in medical data analysis. Ramos *et al.* (2020) provided a comprehensive overview of the Fréchet distribution, a widely used model for extreme value analysis. The authors focus on its statistical properties, parameter estimation techniques, and real-world applications across different fields. Almetwally and Muhammed (2020) contributed to statistical literature by presenting a new bivariate distribution with Fréchet marginals, constructed via copula functions. Alzeley *et al*. (2021) present a well-structured and insightful study that significantly contributes to the field of statistical distribution theory and its applications in reliability and medical research. The NEXF distribution offers a flexible and practical model for analysing complex lifetime data, particularly under censored conditions. Shafiq *et al.* (2021) introduced the Modified KiesFréchet (MKIF) distribution, a three-parameter model combining the properties of the Fréchet and modified Kies distributions. This new distribution is designed to effectively model COVID-19 mortality rates (Shafiq *et al*., 2021), particularly in countries like Canada and the Netherlands (Shafiq *et al.*, 2021). Alshanbari (2022) contributed to the statistical literature by presenting a novel extension of the Fréchet distribution (Alshanbari, 2022), providing both theoretical insights and practical applications that highlight its effectiveness in modelling extreme value data (Alshanbari, 2022).

Existing probability distributions, such as the Rayleigh and Fréchet distributions (Sadiq *et al*., 2024), often fail to accurately model datasets with diverse tail behaviours and skewness patterns (Sadiq *et al.*, 2024). The Rayleigh distribution is limited to light-tailed data, while the Fréchet distribution struggles with varying degrees of skewness and extreme values. This limitation affects the accuracy of statistical modelling in fields like reliability analysis, survival studies, and financial risk assessment. To address this gap, this study proposes the Odd Rayleigh-Fréchet (ORF) distribution, which integrates the strengths of both distributions while introducing additional shape parameters for greater flexibility. However, the mathematical properties, parameter estimation techniques, and real-world applicability of this new model remain unexplored.

This study aims to develop a new Odd Rayleigh-Fréchet (ORF) distribution with its statistical properties and applications. The aim was achieved through the following objectives: by developing the CDF and PDF of the ORF distribution, deriving some of the statistical properties ORF distribution, estimating the parameters of the ORF distribution using the MLE technique, validating the efficiency of the parameters estimated through simulations, and comparing the performance of the ORF distribution with existing models using real datasets (Sadiq *et al*., 2024).

The development of the Odd Rayleigh-Fréchet (ORF) distribution will provide a more flexible statistical model for analysing data with varying tail behaviours, skewness, and extreme values. Many existing distributions, such as the Rayleigh and Fréchet, have limitations in capturing the complexity of real-world datasets in fields like reliability analysis, survival studies, and financial risk assessment. By integrating the strengths of both distributions and introducing additional shape parameters, the ORF distribution will enhance data-fitting accuracy, making it a valuable tool for researchers and practitioners (Sadiq *et al*., 2024). This study will contribute to statistical literature by deriving the mathematical properties, parameter estimation methods, and goodness-of-fit measures for the ORF distribution (Sadiq *et al*., 2024). Through simulations and real-world applications, the research will demonstrate its superiority over existing models, offering better predictive performance (Sadiq *et al*., 2024). The findings will benefit industries such as engineering, medicine, economics, and environmental sciences, where precise statistical modelling is crucial for decision-making and risk assessment (Sadiq *et al*., 2024).

This study focuses on the development, characterisation, and application of the Odd Rayleigh-Fréchet (ORF) distribution as a new probability distribution (Sadiq *et al*., 2024). It covers the derivation of its probability density function (PDF), cumulative distribution function (CDF), and key statistical properties, including moments, skewness, and kurtosis (Sadiq *et al*., 2024). Additionally, the study explores parameter estimation techniques, primarily using the Maximum Likelihood Estimation (MLE) method, and validates the model through simulation studies (Sadiq *et al*., 2024). The applicability of the ORF distribution is assessed using real-world datasets from fields such as survival analysis, reliability engineering, and financial risk modelling, with comparisons made against existing distributions (Sadiq *et al*., 2024).

However, the study has certain limitations. The research primarily focuses on theoretical and empirical validation of the ORF distribution, and while real-world datasets are used, the study does not cover all possible applications. Additionally, the performance of the ORF distribution is evaluated under specific parameter estimation techniques (the MLE), and alternative estimation methods such as Bayesian inference are not explored in depth. Lastly, the study assumes independent and identically distributed (i.i.d.) data, which may not fully account for dependence structures present in some real-world scenarios.

MATERIALS AND METHODS

The Odd Rayleigh-G Family of Distribution

Sadiq *et al*. (2024) defined a random variable X which follows an Odd Rayleigh-G Family of distribution with CDF and PDF (Sadiq *et al*., 2024) given as;

$$F(x;\theta,\xi)=1-\exp\left\{-\frac{1}{2\theta^2}\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^2\right\} \quad (1)$$

where $\theta>0$ is the scale parameter, $x>0$ and $G(x;\xi)$ is the CDF and ξ is the parameters' vector of the baseline distribution (Sadiq *et al*., 2024).

$$f(x;\theta,\xi)=\frac{g(x;\xi)G(x;\xi)}{\theta^2\left(1-G(x;\xi)\right)^3}\exp\left\{-\frac{1}{2\theta^2}\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^2\right\} \quad (2)$$

where $g(x;\xi)$ is the PDF and ξ is the parameters' vector of the baseline distribution (Sadiq *et al*., 2024).

The Frechet Distribution

Ul-Haq and Elgarhy (2018) defined a random variable X which follows the Frechet distribution with CDF and PDF (Ul-Haq & Elgarhy, 2018) given as;

$$G(x;\beta)=\exp\left\{-\left(\frac{1}{x}\right)^{\beta}\right\};\ \forall x,\beta>0 \quad (3)$$

$$g(x;\beta)=\beta x^{-(\beta+1)}\exp\left\{-\left(\frac{1}{x}\right)^{\beta}\right\};\ \forall x,\beta>0 \quad (4)$$

where β is the shape parameter (Ul-Haq & Elgarhy, 2018).

Proposed Odd Rayleigh-Frechet Distribution

Substituting equation (3) into (1) and also substituting equations (3) and (4) into (2), we obtain the CDF and PDF with their respective plots of the new proposed distribution called Odd Rayleigh-Frechet distribution and respectively given as;

$$F(x;\theta,\beta)=1-\exp\left\{-\frac{1}{2\theta^2}\left(\frac{\exp\left\{-\left(\frac{1}{x}\right)^{\beta}\right\}}{1-\exp\left\{-\left(\frac{1}{x}\right)^{\beta}\right\}}\right)^2\right\};\ \forall x,\theta,\beta>0 \quad (5)$$

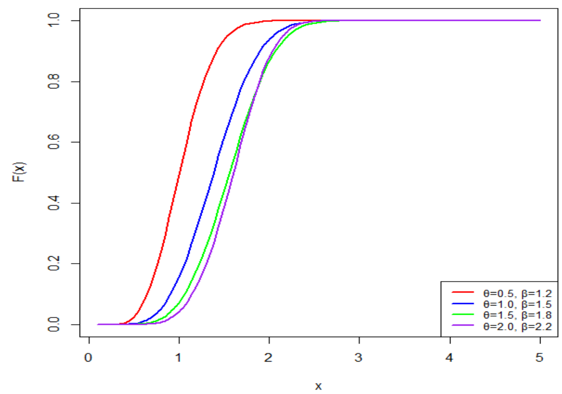

Figure 1: CDF Plot of the Odd Rayleigh-Frechet Distribution

Figure 1 shows how changes in parameters θ and β affect the location and steepness of the CDF. Smaller θ and lower β lead to quicker accumulation of probability at smaller x values, while increasing them spreads the distribution and shifts it rightward. This kind of analysis is useful in model fitting and understanding distribution behaviour (Semary *et al*., 2025).

$$f(x;\theta,\beta)=\frac{\beta x^{-(\beta+1)}\exp\left\{-2\left(\frac{1}{x}\right)^{\beta}\right\}}{\theta^2\left(1-\exp\left\{-\left(\frac{1}{x}\right)^{\beta}\right\}\right)^3}\exp\left\{-\frac{1}{2\theta^2}\left(\frac{\exp\left\{-\left(\frac{1}{x}\right)^{\beta}\right\}}{1-\exp\left\{-\left(\frac{1}{x}\right)^{\beta}\right\}}\right)^2\right\};\ \forall x,\theta,\beta>0 \quad (6),$$

where θ is the scale parameter and β is the shape parameter.

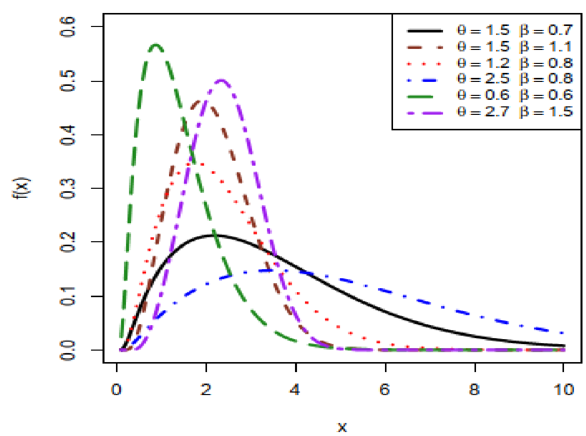

Figure 2: PDF Plot of the Odd Rayleigh-Frechet Distribution

Figure 3.2 illustrates that when β is small (e.g., 0.7 or 0.8), the curve is right-skewed, meaning it has a long tail to the right (Sadiq *et al*., 2023b). As β increases (e.g., 1.1, 1.4, 1.5), the distribution becomes more symmetric and the peak moves rightward. θ affects the spread; the smaller θ (like 0.6) the curve is narrow and tall (values cluster near small x). Larger θ (like 2.7), the curve spreads out and the peak shifts to the right. These density behaviours help in understanding how the model responds to different parameter settings and are useful for fitting real-life data with appropriate skewness or tail behaviour (Mohammed *et al*., 2025).

Survival and Hazard Function of Odd Rayleigh-Frechet Distribution

The survival and hazard function (Sadiq *et al.*, 2023c) of the Odd Rayleigh-Frechet Distribution are respectively obtained as;

$$S(x;\theta,\beta)=\exp\left\{-\frac{1}{2\theta^2}\left(\frac{\exp\left\{-\left(\frac{1}{x}\right)^{\beta}\right\}}{1-\exp\left\{-\left(\frac{1}{x}\right)^{\beta}\right\}}\right)^2\right\};\forall x,\theta,\beta>0 \quad (7)$$

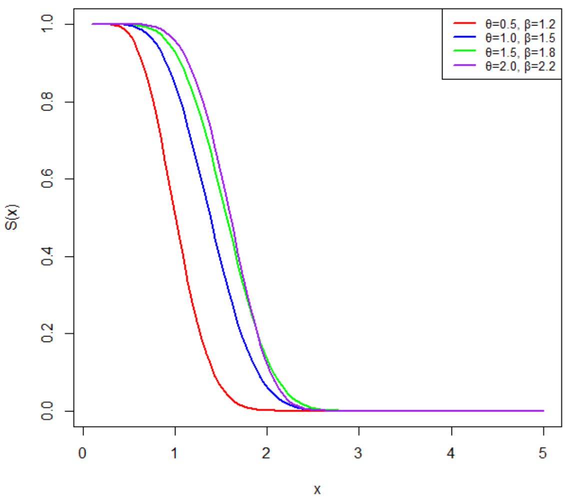

Figure 3: Survival Function Plot of the Odd Rayleigh-Frechet Distribution

Figure 3 demonstrates the survival behaviour of a distribution under different parameters. Larger values of θ and βimply longer survival and heavier tails. Smaller values of θ and β imply quicker decay, shorter survival, useful in modelling systems, machines, or lifetimes where predicting how long something lasts matters (Oga *et al*., 2025).

$$h(x;\theta,\beta)=\frac{\beta x^{-(\beta+1)}\exp\left\{-2\left(\frac{1}{x}\right)^{\beta}\right\}}{\theta^2\left(1-\exp\left\{-\left(\frac{1}{x}\right)^{\beta}\right\}\right)^3};\ \forall x,\theta,\beta>0 \quad (8)$$

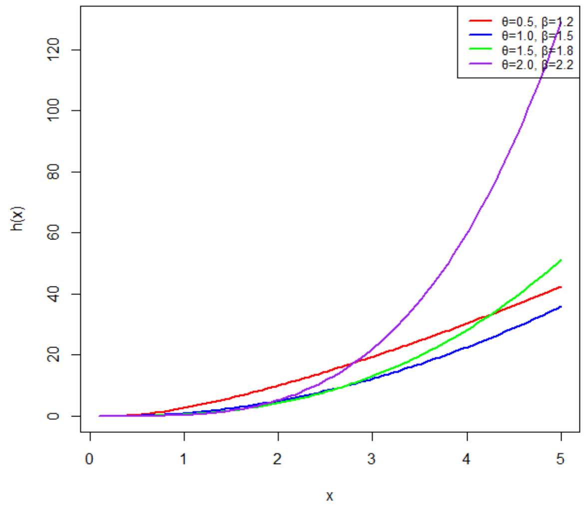

Figure 4: Hazard Function Plot of the Odd Rayleigh-Frechet Distribution

Figure 4 shows that the distribution becomes heavier-tailed and more spread out as θ increases. Higher value of β leads to lighter tails, more concentration, and sometimes loss of higher value of moments (e.g., all zero for θ = 0.5, β = 3.0). The results facilitate calibration of the distribution to match the specific skewness, tail weight, or variability characteristics of real-world data (Abd Elgawad *et al.*, 2025).

Quantile Function of OR-F Distribution

The quantile function $Q(u)$ is derived by solving the cumulative distribution function (CDF) for x in terms of u as (Sadiq *et al.*, 2023b):

$$x=Q(u;\theta,\beta)=\left(-\ln\left(\frac{\sqrt{-2\theta^2\ln(1-u)}}{\left(1+\sqrt{-2\theta^2\ln(1-u)}\right)}\right)\right)^{-\frac{1}{\beta}} \quad (9)$$

This function in equation (9) can be used to generate random numbers (Sadiq *et al.*, 2023b) from the Odd Rayleigh-Fréchet (OR-F) distribution by substituting a uniform random variable (Sadiq *et al.*, 2023c) $u\sim U(0,1)$ into $Q(u)$.

Moments of OR-F Distribution

To derive the moments of the Odd Rayleigh-Fréchet (OR-F) distribution, we calculate the expected value of X^r, given by:

$$E[X^r]\approx\frac{\Gamma\left(\frac{r}{\beta}+1\right)}{\theta^2}\times\int_0^{\infty}\frac{\exp(-2t)}{(1-\exp(-t))^3}\exp\left\{-\frac{1}{2\theta^2}\left(\frac{\exp(-t)}{1-\exp(-t)}\right)^2\right\}dt \quad (10)$$

This form suggests that moments of the OR-F distribution can be approximated using the Gamma function properties and numerical integration.

Table 1: Computed moments for different parameter values

Theta	Beta	Moment_Order	Moment_Value
0.5	1.0	1	1.0376
0.5	1.0	2	1.2046
0.5	1.0	3	1.5373
0.5	1.0	4	2.1268
0.5	2.0	1	1.0032
0.5	2.0	2	1.0376
0.5	2.0	3	1.1038
0.5	2.0	4	1.2046
0.5	3.0	1	0.0000
0.5	3.0	2	0.0000
0.5	3.0	3	0.0000
0.5	3.0	4	0.0000
1.0	1.0	1	1.6949
1.0	1.0	2	3.3341
1.0	1.0	3	7.3893
1.0	1.0	4	18.0609
1.0	2.0	1	1.2750
1.0	2.0	2	1.6949
1.0	2.0	3	2.3382
1.0	2.0	4	3.3341
1.0	3.0	1	1.1701
1.0	3.0	2	1.3958
1.0	3.0	3	1.6949
1.0	3.0	4	2.0923
1.5	1.0	1	2.3359
1.5	1.0	2	6.4618
1.5	1.0	3	20.3819
1.5	1.0	4	71.4417
1.5	2.0	1	1.4919
1.5	2.0	2	2.3359
1.5	2.0	3	3.8141
1.5	2.0	4	6.4618
1.5	3.0	1	1.2982
1.5	3.0	2	1.7237
1.5	3.0	3	2.3359
1.5	3.0	4	3.2250
2.0	1.0	1	2.9710
2.0	1.0	2	10.5889
2.0	1.0	3	43.3346
2.0	1.0	4	197.9089
2.0	2.0	1	1.6789
2.0	2.0	2	2.9710
2.0	2.0	3	5.5002
2.0	2.0	4	10.5889
2.0	3.0	1	1.4038
2.0	3.0	2	2.0197
2.0	3.0	3	2.9710
2.0	3.0	4	4.4587

Table 1 presents the computed moments (Moment_Value) of aOR-F distribution for different values of θ and β across different Moment Orders.

The effect of the scale parameter (θ) on the moments, as θ increases, the moments tend to increase significantly, especially for higher moment orders. For example, when $\theta = 0.5$, the 4th moment ranges from 1.2046 to 2.1268, but when $\theta = 2.0$, the 4th moment goes up to 197.9089 (for $\beta = 1.0$). This suggests that higher values of θ lead to a heavier tail and greater dispersion in the distribution. The effect of the shape parameter (β) on the moments, increasing β generally results in lower moment values(except for some cases). For example, when $\theta = 1.5$ increasing β from 1.0 to 3.0 leads to a decline in the moment of order 4 from 71.4417 ($\beta = 1.0$) to 3.2250 ($\beta = 3.0$). This suggests that higher values of β make the distribution less dispersed and may lead to a more concentrated shape. This analysis can help in parameter selection for fitting the distribution to real-world data, particularly in finance, survival analysis, or heavy-tailed phenomena.

Moment Generating Function of OR-F Distribution (MGF)

To derive the Moment-Generating Function (Sadiq *et al*., 2023a) of the Odd Rayleigh-Fréchet (OR-F) distribution, we use the definition of the MGF:

$$M_X(t) \approx \frac{1}{\theta^2} \sum_{n=0}^{\infty} \frac{t^n}{n!} \Gamma\left(\frac{n}{\beta} + 1\right) \qquad (11)$$

This equation (11) suggests that the MGF can be expressed in terms of the Gamma function, but it does not yield a simple closed-form expression.

Table 2: MGF values of OR-F distribution for different values of t, θ, and β

θ	β	t	MGF_Value
0.50	1.00	-0.50	0.6046

0.50	1.00	0.00	1.0000
0.50	1.00	0.50	1.7079
0.50	1.00	1.00	3.0201
0.50	2.00	-0.50	0.6079
0.50	2.00	0.00	1.0000
0.50	2.00	0.50	1.6578
0.50	2.00	1.00	2.7699
0.50	3.00	-0.50	0.0000
0.50	3.00	0.00	0.0000
0.50	3.00	0.50	0.0000
0.50	3.00	1.00	0.0000
1.00	1.00	-0.50	0.4523
1.00	1.00	0.00	1.0000
1.00	1.00	0.50	2.4815
1.00	1.00	1.00	7.0690
1.00	2.00	-0.50	0.5332
1.00	2.00	0.00	1.0000
1.00	2.00	0.50	1.9082
1.00	2.00	1.00	3.7050
1.00	3.00	-0.50	0.5590
1.00	3.00	0.00	1.0000
1.00	3.00	0.50	1.8010
1.00	3.00	1.00	3.2651
1.50	1.00	-0.50	0.3486
1.50	1.00	0.00	1.0000
1.50	1.00	0.50	3.6927
1.50	1.00	1.00	18.9550
1.50	2.00	-0.50	0.4808
1.50	2.00	0.00	1.0000
1.50	2.00	0.50	2.1377
1.50	2.00	1.00	4.6970
1.50	3.00	-0.50	0.5250
1.50	3.00	0.00	1.0000
1.50	3.00	0.50	1.9230
1.50	3.00	1.00	3.7327
2.00	1.00	-0.50	0.2746
2.00	1.00	0.00	1.0000
2.00	1.00	0.50	5.6766
2.00	1.00	1.00	60.2685
2.00	2.00	-0.50	0.4402
2.00	2.00	0.00	1.0000
2.00	2.00	0.50	2.3596
2.00	2.00	1.00	5.7829
2.00	3.00	-0.50	0.4987
2.00	3.00	0.00	1.0000
2.00	3.00	0.50	2.0299
2.00	3.00	1.00	4.1701

Table 2 provides values of the moment generating function (MGF) for the OR-F distribution under different values of θ, β, and t (moment argument). As expected, the MGF value at $t = 0$ is consistently 1 for all parameter values, confirming that $\mathrm{MGF}(0) = 1$ this is a fundamental property of MGFs.For positive values of t (e.g., $t = 0.5, 1.0$), the MGF values increase, sometimes dramatically. This growth is much steeper for higher θ values. For example, when $\theta = 2.0, \beta = 1.0, t = 1.0$, the MGF is 60.2685, compared to 3.0201 when $\theta = 0.5$. This suggests that higher θ values lead to a more pronounced exponential tail in the distribution.
The negative t values reduce MGF, for $t = -0.5$, the MGF values decrease significantly. The reduction is more drastic for larger θ. For instance, at $\theta = 2.0, \beta = 1.0, t = -0.5$, the MGF is only 0.2746, while for $\theta = 0.5$, it is 0.6046. This indicates that negative moments are suppressed more as θ increases.

Rényi Entropy for the Odd Rayleigh-Fréchet (OR-F) Distribution

The entropy of a probability distribution measures the amount of uncertainty or randomness in the distribution (Sadiq *et al*., 2023b). The Rényi entropy is a generalisation of Shannon entropy, and for the OR-F distribution is derived as:

$$I(\alpha) = \int_0^{\infty} \frac{\beta^{\alpha} x^{-\alpha(\beta+1)} \exp\left\{-2\alpha\left(\frac{1}{x}\right)^{\beta}\right\}}{\theta^{2\alpha}\left(1-\exp\left\{-\left(\frac{1}{x}\right)^{\beta}\right\}\right)^{3\alpha}} \exp\left\{-\frac{\alpha}{2\theta^2}\left(\frac{\exp\left\{-\left(\frac{1}{x}\right)^{\beta}\right\}}{1-\exp\left\{-\left(\frac{1}{x}\right)^{\beta}\right\}}\right)^2\right\} dx \tag{12}$$

The Rényi entropy is computed as:

$$H_{\alpha}(X) = \frac{1}{1-\alpha} \log I(\alpha) \tag{13}$$

Since the integral is difficult to solve analytically, we compute it numerically using R.

Table 3: Computed Rényientropy value of OR-F distribution for different θ, β, α

Alpha	Theta	Beta	Renyi_Entropy
0.5	0.5	1	0.531112
1.5	0.5	1	0.281495
2.0	0.5	1	0.228743
0.5	0.5	2	-0.155514

1.5	0.5	2	-0.397678
2.0	0.5	2	-0.450434
0.5	0.5	3	-0.552148
1.5	0.5	3	73.39688
2.0	0.5	3	14.75549
0.5	1.0	1	1.159459
1.5	1.0	1	0.912751
2.0	1.0	1	0.860848
0.5	1.0	2	0.234933
1.5	1.0	2	0.00123
2.0	1.0	2	-0.050324
0.5	1.0	3	-0.238513
1.5	1.0	3	-0.480669
2.0	1.0	3	17.528067
0.5	1.5	1	1.542584
1.5	1.5	1	1.297854
2.0	1.5	1	1.246463
0.5	1.5	2	0.463415
1.5	1.5	2	0.233247
2.0	1.5	2	0.182087
0.5	1.5	3	-0.059988
1.5	1.5	3	-0.30035
2.0	1.5	3	19.149926
0.5	2.0	1	1.81929
1.5	2.0	1	1.575969
2.0	2.0	1	1.524918
0.5	2.0	2	0.624288
1.5	2.0	2	0.395811
2.0	2.0	2	0.344769
0.5	2.0	3	0.063415
1.5	2.0	3	30.261136
2.0	2.0	3	20.300653

Table 3 presents the computedRényi entropy values for different parameter combinations of the OR-F distribution. The results provide insights into how the entropy changes with variations in $\alpha, \theta, and \beta$. The entropy generally decreases as α increases, suggesting that higher values of α lead to more concentrated distributions. Negative entropy values appear in some cases ($\beta = 2, 3$), which might indicate numerical instability or specific properties of the OR-F distribution. Some extreme values (e.g., 73.39688 and 30.261136) could suggest high sensitivity to parameter variations.

Order Statistics of the Odd Rayleigh-Fréchet (OR-F) Distribution

Order statistics deal with the properties of ranked observations from a sample (Sadiq *et al.*, 2023c). Given a random sample $X_1, X_2, \ldots, X_n$ from the OR-F distribution, the order statistics are denoted as :

$$X_{(1)} \le X_{(2)} \le \ldots \le X_{(n)} \quad (14)$$

where $X_{\{(k)\}}$ represents the kth order statistic, meaning the kth smallest value in the sample (Sadiq *et al.*, 2023b).
The PDF of the kth order statistic from a sample of size n is given by:

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} F(x)^{k-1} [1-F(x)]^{n-k} f(x) \quad (15)$$

where $f(x;\theta,\beta)$ and $F(x;\theta,\beta)$ are the PDF and CDF of the OR-F distribution presented in equations (6) and (5) respectively, n is the sample size and k is the rank of the order statistic.
The CDF of the kth order statistic is:

$$F_{X_{(k)}}(x) = \sum_{j=k}^{n} \binom{n}{j} F(x)^j [1-F(x)]^{n-j} \quad (16)$$

This expression gives the probability that the kth smallest value is less than or equal to x.
The expectation of the k-th order statistic is given by:

$$E[X_{(k)}] = n\int_0^{\infty} x f(x) F(x)^{k-1} [1-F(x)]^{n-k} \, dx \quad (17)$$

which generally requires numerical integration for the OR-F distribution.
Substituting equations (5) and (6) into the general formula for order statistics presented in equation (15), we obtain:

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} \left(1-e^{-(1/x)^{\beta}}\right)^{k-1} e^{-(n-k)(1/x)^{\beta}} f(x) \quad (18)$$

which gives the distribution of the kth smallest value in an OR-F-distributed sample.

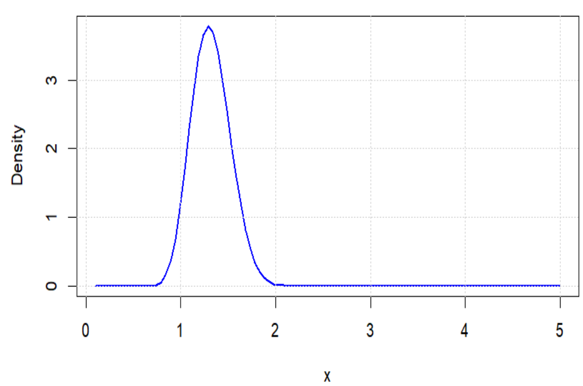

Figure 5: PDF of 5th order statistic in OR-F distribution

Log-Likelihood Function for the Odd Rayleigh-Fréchet (OR-F) Distribution

The log-likelihood function is derived using the probability density function (Sadiq *et al.*, 2023c) of the Odd Rayleigh-Fréchet (OR-F) distribution. Let $x_1, x_2, \ldots, x_n$ be an independent and identically distributed (i.i.d.) sample (Sadiq *et al.*, 2023a) from the OR-F distribution. The log-likelihood function for the OR-F distribution.

$$\log L(\theta,\beta) = n\log\beta - (\beta+1)\sum_{i=1}^{n}\log x_i$$
$$-2\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^{\beta} - 2n\log\theta - \qquad (19)$$
$$3\sum_{i=1}^{n}\log\left(1-e^{-(1/x_i)^{\beta}}\right) - \frac{1}{2\theta^2}\sum_{i=1}^{n}\left(\frac{e^{-(1/x_i)^{\beta}}}{1-e^{-(1/x_i)^{\beta}}}\right)^2$$

We now compute the partial derivatives of the log-likelihood function (Sadiq *et al.*, 2023b) $\log L(\theta,\beta)$ presented in equation (19) with respect to θ and β.

$$\frac{\partial \log L}{\partial \theta} = -\frac{2n}{\theta} + \frac{1}{\theta^3}\sum_{i=1}^{n}\left(\frac{e^{-(1/x_i)^{\beta}}}{1-e^{-(1/x_i)^{\beta}}}\right)^2 \qquad (20)$$

Setting equation (20) to zero for MLE:

$$\frac{2n}{\theta} + \frac{1}{\theta^3}\sum_{i=1}^{n}\left(\frac{e^{-(1/x_i)^{\beta}}}{1-e^{-(1/x_i)^{\beta}}}\right)^2 = 0 \qquad (21)$$

$$\frac{\partial \log L}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n}\log x_i -$$
$$2\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^{\beta}\log\left(\frac{1}{x_i}\right) -$$
$$3\sum_{i=1}^{n}\frac{e^{-(1/x_i)^{\beta}}\log\left(\frac{1}{x_i}\right)}{1-e^{-(1/x_i)^{\beta}}} \qquad (22)$$
$$-\frac{1}{2\theta^2}\sum_{i=1}^{n}\left(\frac{2e^{-(1/x_i)^{\beta}}\log(1/x_i)\cdot(1-e^{-(1/x_i)^{\beta}}) + e^{-2(1/x_i)^{\beta}}\log(1/x_i)}{(1-e^{-(1/x_i)^{\beta}})^3}\right)$$

Setting equation (22) to zero for MLE**:**

$$\frac{n}{\beta} - \sum_{i=1}^{n}\log x_i - 2\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^{\beta}\log\left(\frac{1}{x_i}\right)$$
$$-3\sum_{i=1}^{n}\frac{e^{-(1/x_i)^{\beta}}\log\left(\frac{1}{x_i}\right)}{1-e^{-(1/x_i)^{\beta}}} \qquad (23)$$
$$-\frac{1}{2\theta^2}\sum_{i=1}^{n}\left(\frac{2e^{-(1/x_i)^{\beta}}\log(1/x_i)\cdot(1-e^{-(1/x_i)^{\beta}}) + e^{-2(1/x_i)^{\beta}}\log(1/x_i)}{(1-e^{-(1/x_i)^{\beta}})^3}\right) = 0$$

Equations (21`) and (23)are nonlinear; they can only be solved numerically.

Simulation Procedure

Step-by-step mathematical algorithm for estimating the parameters of the ORF distribution using Monte Carlo Simulation (Dangana *et al.*, 2026a; Dangana *et al.*, 2026b).

Step 1: Generate Random Samples

The quantile function provided is:

$$x = Q(u;\theta,\beta) = \left(-\ln\left(\frac{\sqrt{-2\theta^2\ln(1-u)}}{\left(1+\sqrt{-2\theta^2\ln(1-u)}\right)}\right)\right)^{-\frac{1}{\beta}}$$

Where u is a uniformly distributed random number (Dangana *et al*., 2026b) in the interval (0, 1).

Step 2: Maximum Likelihood Estimation (MLE)

The log-likelihood function for $\boldsymbol{n}$ samples is:

$$\log L(\beta,\theta) = \sum_{i=1}^{n}\log f(x;\beta,\theta)$$

Step 3: Monte Carlo Simulation

a. Generate Random Samples:

Use the quantile function to generate n samples $x_1, x_2, \ldots, x_n$ from the distribution.

b. Estimate Parameters Using MLE:

I. Define the log-likelihood function.

II. Use numerical optimization (Newton-Raphson Method) to find the parameter estimates that maximize the log-likelihood function.

c. Repeat Simulation:

I. Repeat steps 1 and 2 for N iterations to get multiple sets of parameter estimates.

d. Compute Mean and Root Mean Square Error (RMSE):

I. Calculate the mean and RMSE of the estimated parameters over the N iterations.

Step 4: Mean and Root Mean Square Error (RMSE) Calculation:

i. For parameter α
ii. The mean of estimates:

$$Mean\left(\hat{\alpha}\right)=\frac{1}{N}\sum_{i=1}^{N}\hat{\alpha}_i$$

iii. Bias: $Bias\left(\hat{\alpha}\right)=\left(\hat{\alpha}_i-\alpha\right)$
iv. RMSE of estimates:

$$RMSE\left(\hat{\alpha}\right)=\sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\hat{\alpha}_i-\alpha\right)^2}$$

The Mean and RMSE calculation for all other parameters follows the same pattern.

Performance Assessment

To assess the performance of the developed model, the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC) are employed (Mohammed et al., 2025) in this study.

RESULTS AND DISCUSSION

Simulation Study

To evaluate whether the parameter estimates of the developed distributions were consistent (Sadiq *et al*., 2023b), a simulation study was carried out using the Monte Carlo simulation technique by computing the means, bias, and root mean square error of the estimated parameters from the maximum likelihood estimation method (Sadiq *et al*., 2023b). The quantile function in equation (9) for the ORF distribution was used to generate the simulated data for different sample sizes n = 20, 50, 100, 200, 300, 400, 500 and 1000, with each sample replicated 1000 times.

Table 4: Simulation Results for the $\theta=1.5$ and $\beta=2.0$ of ORFD

Sample_Size	Parameters	Estimates	Bias	RMSE
20	β	1.551179	-0.4488209	0.7998634
	θ	2.519482	1.0194821	3.3448354
50	β	1.556995	-0.4430053	0.6227184
	θ	1.792707	0.2927072	2.2596834
100	β	1.588164	-0.4118357	0.5001812
	θ	1.372139	-0.1278606	1.2777127
200	β	1.598814	-0.4011857	0.4276745
	θ	1.182479	-0.3175205	0.3320648
300	β	1.593385	-0.4066152	0.4259361
	θ	1.181945	-0.3180551	0.3289656
400	β	1.596828	-0.4031724	0.4155378
	θ	1.175072	-0.3249281	0.3312032
500	β	1.593036	-0.406964	0.4180031
	θ	1.178025	-0.3219751	0.3277642
1000	β	1.595995	-0.4040051	0.409515
	θ	1.17205	-0.3279495	0.3306509

Table 4 presents the simulation results for parameter estimation of the OR-F distribution. The bias for both parameters (β and θ) decreases as the sample size increases, indicating improved estimation accuracy. For small samples ($n=20, n=50$), the bias is relatively large, especially for θ, suggesting that the estimates are further from the true values. As n increases to 1000, the bias stabilises and becomes smaller. The RMSE decreases with increasing sample size, meaning that larger datasets yield more precise estimates. For small n, RMSE is high (e.g., RMSE of θ at $n=20$ is 3.3448), but it drops significantly as n increases, reaching around 0.33 at $n=1000$. This confirms the consistency of the estimators; as the sample size grows, the accuracy of estimation improves. The estimates for β appear more stable, with a steady decrease in bias and RMSE.The θ estimates show higher bias and RMSE at small n, indicating more variability in its estimation. The estimators for β and θ improve with larger sample sizes, as seen in the declining bias and RMSE.Small sample sizes lead to high bias and RMSE, making estimation less reliable. For $n \geq 200$ the estimators perform well, with reduced bias and RMSE, suggesting that a minimum of 200 observations is recommended for reliable estimation.

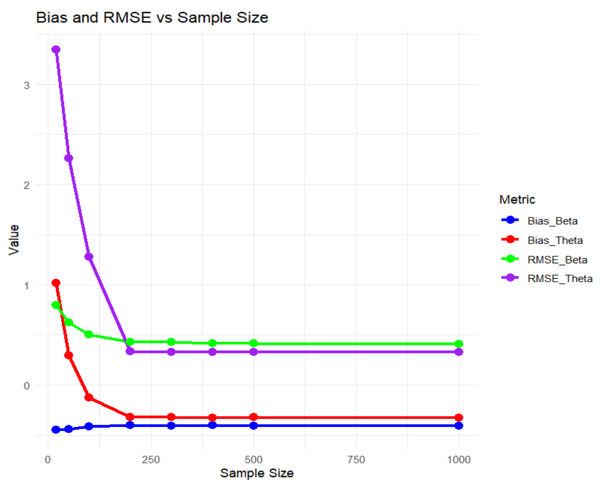

Figure 6: Biases and RMSE for the Simulation

Figure 6 illustrates the relationship between sample size and the bias and RMSE of the parameter estimates (β and θ). As the sample size increases, both bias and RMSE decrease (Sadiq *et al.*, 2023c), indicating improved estimation accuracy and consistency. The bias for θ starts higher but declines sharply, while β exhibits a smoother reduction (Sadiq *et al.*, 2023b). RMSE follows a similar trend, with θ initially having a larger error that stabilises after $n \geq 200$. generally, the estimators are asymptotically unbiased and consistent, with a sample size of at least 200 recommended for reliable estimation.

Table 5: Simulation Results for the $\theta = 1.6$ and $\beta = 2.1$ of ORFD

Sample_Size	Parameters	Estimates	Bias	RMSE
20	β	1.394972	-0.7050283	0.9373734
	θ	2.933123	1.3331233	3.6688644
50	β	1.406785	-0.6932146	0.8217602
	θ	2.216221	0.6162209	2.7283020
100	β	1.443563	-0.6564374	0.7279625
	θ	1.737844	0.1378441	1.8664291
200	β	1.474164	-0.2482601	0.5478671
	θ	1.351740	-0.3175205	0.3320648
300	β	1.470357	-0.6296431	0.6433403
	θ	1.350088	-0.2499121	0.5450738
400	β	1.476612	-0.6233880	0.6303265
	θ	1.316909	-0.2830914	0.2935052
500	β	1.473284	-0.6267165	0.6329226
	θ	1.320073	-0.2799273	0.2895675
1000	β	1.476804	-0.6231958	0.6262353
	θ	1.312573	-0.2874274	0.2917481

Table 5 presents the simulation results for parameter estimation of the OR-F distribution. The ORFD model shows a clear improvement in estimation accuracy as the sample size increases, particularly for the parameter θ. For small samples ($n = 20 - 50$), both β and θ exhibit relatively large biases and RMSE values, indicating substantial variability and deviation from their true values. As the sample size grows, the RMSE values steadily decline, reflecting more stable and reliable estimates. The bias of θ decreases markedly and approaches zero around $n = 100 - 500$, demonstrating strong consistency, while the bias for β remains negative and relatively stable across sample sizes, suggesting a systematic underestimation that persists even in larger samples. Finally, the pattern indicates that the estimator for θ performs increasingly well with larger samples, whereas the estimator for β shows improved precision but maintains a consistent downward bias as depicted in Figure 4.2, highlighting the need for caution when interpreting β estimates in small to moderate samples.

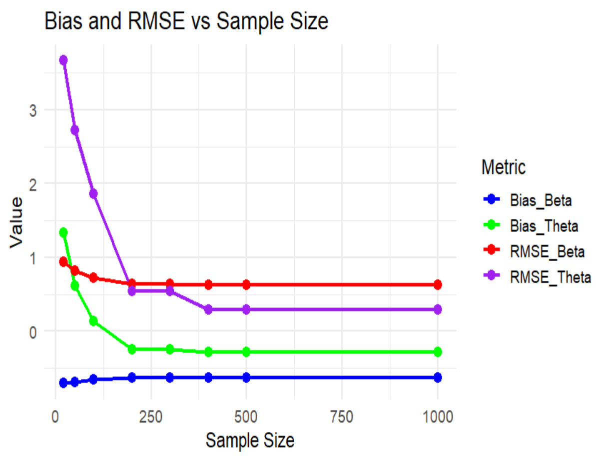

Figure 7: Biases and RMSE for the Simulation

Applications to Real-life Datasets

We used some of the existing datasets to compare the performance of the proposed distribution and other related distributions. The competing models are the Frechet distribution and the Weibull distribution. The data sets are obtained from Ramos *et al*. (2019), which represent the total monthly rainfall during May, June, July, August and September at Piracicaba River (Ramos *et al*., 2019). The Piracicaba River is a river located in Brazil, primarily in the state of São Paulo (Ramos *et al*., 2019). The Piracicaba River is about 250 kilometres (155 miles) long (Ramos *et al*., 2019). The datasets are:

"**May**: 29.19, 18.47, 12.86, 151.11, 19.46, 19.46, 84.30, 19.30, 18.47, 34.12, 374.54, 19.72, 25.58, 45.74, 68.53, 36.04, 15.92, 21.89, 40.00, 44.10, 33.35, 35.49, 56.25, 24.29, 23.56, 50.85, 24.53, 13.74, 27.99, 59.27, 13.31, 41.63, 10.00, 33.62, 32.90, 27.55, 16.76, 47.00, 106.33, 21.03 (Ramos *et al*., 2019).

June: 13.64, 39.32, 10.66, 224.07, 40.90, 22.22, 14.44, 23.59, 47.02, 37.01, 432.11, 10.63, 28.51, 11.77, 25.35, 25.80, 39.73, 9.21, 22.36, 11.63, 33.35, 18.00, 18.62, 17.71, 100.10, 23.32, 11.63, 10.20, 12.04, 11.63, 50.57, 11.63, 33.72, 14.69, 12.30, 32.90, 179.75, 37.57, 7.95 (Ramos *et al*., 2019).

July: 12.98, 15.66, 13.18, 174.94, 10.35, 47.52, 13.28, 24.03, 11.40, 22.71, 43.96, 9.38, 11.40, 13.28, 14.84, 14.44, 63.74, 12.04, 17.26, 28.74, 12.25, 10.22, 26.25, 13.31, 28.24, 12.88, 17.71, 8.82, 10.40, 7.67, 49.15, 17.93, 9.80, 105.88, 10.77, 13.49, 19.77, 34.22, 7.26 (Ramos *et al*., 2019).

August: 16.00, 9.52, 9.43, 53.72, 17.10, 8.52, 10.00, 15.23, 8.78, 28.97, 28.06, 18.26, 9.69, 51.43, 10.96, 13.74, 20.01, 10.00, 12.46, 10.40, 26.99, 7.72, 11.84, 18.39, 11.22, 13.10, 16.58, 12.46, 58.98, 7.11, 11.63, 8.24, 9.80, 15.51, 37.86, 30.20, 8.93, 14.29, 12.98, 12.01, 6.80 (Ramos *et al*., 2019).

September: 29.19, 8.49, 7.37, 82.93, 44.18, 13.82, 22.28, 28.06, 6.84, 12.14, 153.78, 17.04, 13.47, 15.43, 30.36, 6.91, 22.12, 35.45, 44.66, 95.81, 6.18, 10.00, 58.39, 24.05, 17.03, 38.65, 47.17, 27.99, 11.84, 9.60, 6.72, 13.74, 14.60, 9.65, 10.39, 60.14, 15.51, 14.69, 16.44 (Ramos *et al*., 2019)".

The rainfall datasets used in this study were previously analysed by Ramos *et al*. (2019) to assess the flexibility of their proposed Fréchet model. In the present work, we have newly developed a model extension of the Fréchet distribution. For purposes of methodological consistency and to provide a justified basis for comparison, we employ the same dataset used by Ramos *et al*. (2019) without modification. This enables a direct and focused evaluation of the performance and flexibility of our proposed distribution relative to the competing models in their study.

Table 6: Goodness of Fit Measures for Total Rainfall in May at Piracicaba River

Model	θ	β	LogLikelihood	AIC	BIC
ORFD	44.5100	9.8145	-46.6	97.2	101
FD	115.1070	0.4821	-126	256	259
WD	41.0159	1.5330	-192	388	391

Table 6 presents the goodness-of-fit of various models for total rainfall in May at the Piracicaba River, comparing their parameter estimates (β and θ), log-likelihood, Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC). The goodness-of-fit table compares the proposed Odd Rayleigh Fréchet Distribution (ORFD) with the competing models: Fréchet Distribution (FD) and Weibull Distribution (WD) for total rainfall in May at the Piracicaba River. ORFD achieves the lowest AIC (97.2) and BIC (101), indicating a superior model fit compared to FD and WD, which have significantly higher AIC and BIC values. These results suggest that ORFD provides the best goodness-of-fit, making it the most suitable model for this dataset.

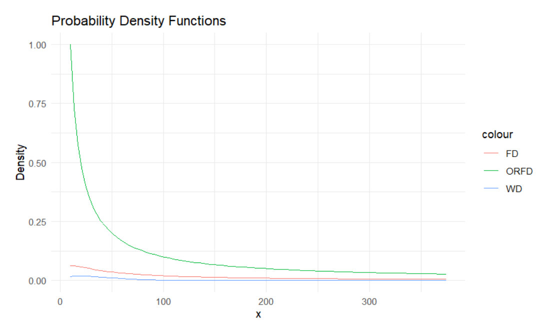

Figure 8: Density Plot for Total Rainfall in May at Piracicaba River

Table 7: Goodness of Fit Measures for Total Rainfall in June at Piracicaba River

Model	θ	β	LogLikelihood	AIC	BIC
ORFD	37.2900	10.1240	-35.7	75.4	78.7
FD	84.9071	0.5120	-120	244	247
WD	40.9493	0.8904	-186	376	380

Table 7 presents the goodness-of-fit results for total rainfall in June at the Piracicaba River, showing that the Odd Rayleigh Fréchet Distribution (ORFD) outperforms the competing models (Fréchet Distribution (FD) and Weibull Distribution (WD). ORFD achieves the lowest AIC (75.4) and BIC (78.7), indicating a superior fit compared to FD and WD, which have significantly higher values. These results confirm that ORFD provides the best balance between goodness-of-fit and model complexity, making it the most suitable distribution for modelling June rainfall data.

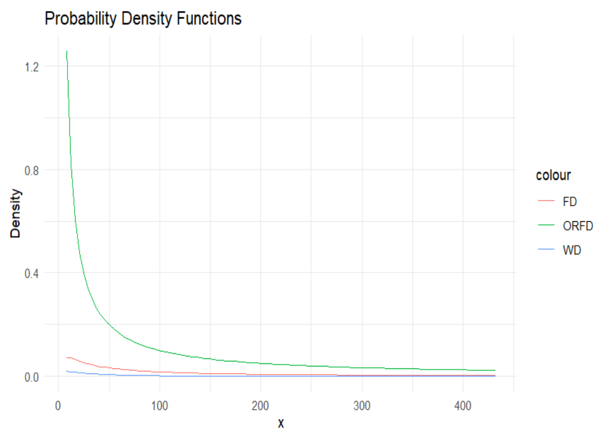

Figure 9: Density Plot for Total Rainfall in June at Piracicaba River

Table 8: Goodness of Fit Measures for Total Rainfall in July at Piracicaba River

Model	θ	β	LogLikelihood	AIC	BIC
ORFD	25.1400	10.1023	-23.3	50.5	53.8
FD	65.2994	0.4910	-112	227	231
WD	26.8584	1.1283	-165	333	336

Table 8 presents the goodness-of-fit results for total rainfall in July at the Piracicaba River, indicating that the Odd Rayleigh Fréchet Distribution (ORFD) provides the best fit among the competing models (Fréchet Distribution (FD) and Weibull Distribution (WD). ORFD achieves the lowest AIC (50.5) and BIC (53.8), confirming its superior performance. FD and WD have significantly higher AIC and BIC values, indicating a poorer fit. These findings reinforce that ORFD is the most appropriate model for describing July rainfall data at the Piracicaba River.

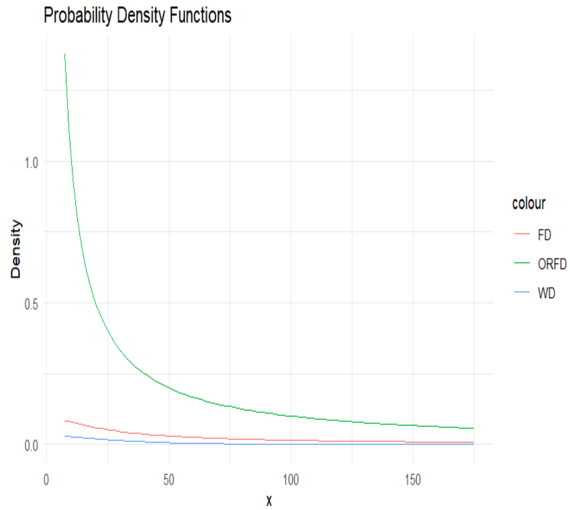

Figure 10: Density Plot for Total Rainfall in July at Piracicaba River

Table 9: Goodness of Fit Measures for Total Rainfall in August at Piracicaba River

Model	θ	β	LogLikelihood	AIC	BIC
ORFD	18.2601	10.0000	-15.5	35.1	38.5
FD	54.5911	0.5000	-111	227	230
WD	19.6764	1.5820	-151	307	310

Table 9 presents the goodness-of-fit results for total rainfall in August at the Piracicaba River, confirming that the Odd Rayleigh Fréchet Distribution (ORFD) is the best-performing model among the tested distributions (Fréchet Distribution (FD), and Weibull Distribution (WD). ORFD achieves the lowest AIC (35.1) and BIC (38.5), indicating a superior fit to the data. FD and WD show significantly higher AIC and BIC values, suggesting weaker fits. These findings reinforce that ORFD provides the most reliable representation of rainfall patterns in August at the Piracicaba River.

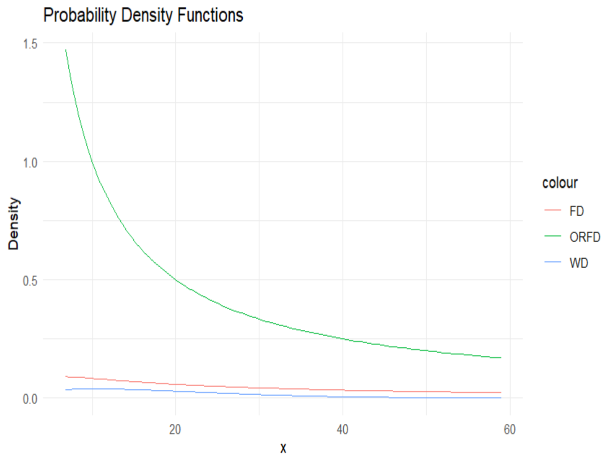

Figure 11: Density Plot for Total Rainfall in August at Piracicaba River

Table 10: Goodness of Fit Measures for Total Rainfall in September at Piracicaba River

Model	θ	β	LogLikelihood	AIC	BIC
ORFD	32.9050	10.4011	-26.9	57.8	61.1
FD	69.0846	0.53107	-115	234	237
WD	30.1775	1.1735	-168	341	344

Table 10 presents the goodness-of-fit results for total rainfall in September at the Piracicaba River, indicating that the Odd Rayleigh Fréchet Distribution (ORFD) provides the best fit among the tested models (Fréchet Distribution (FD) and Weibull Distribution (WD). ORFD achieves the lowest AIC (57.8) and BIC (61.1), reinforcing its superior performance. In contrast, FD and WD display significantly higher AIC and BIC values, indicating poorer model fits. These results further validate ORFD as the most appropriate distribution for modelling rainfall patterns in September at the Piracicaba River.

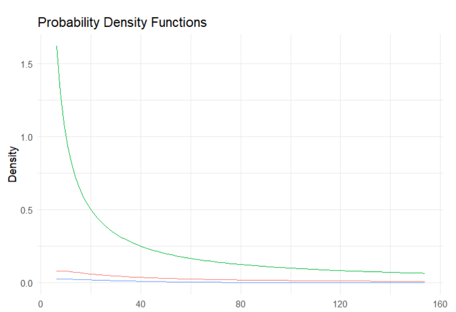

Figure 12: Density Plot for Total Rainfall in September at Piracicaba River

Nigerian Rainfall Dataset

The dataset comprising the values “1111.34, 2230.44, 958.944, 2387.98, 1957.06, 872.661, 2247.54, 1557.22, 629.081, 2138.67, 2099.18, 1961.05, 1850.77, 1499.56, 1818.53, 1225.32, 871.791, 2171.87, 687.822, 1092.98, 890.993, 734.096, 798.441, 1481.02, 1196.11, 1460.72, 1240.24, 1340.17, 1680.74, 1453.87, 1240.48, 1149, 2305.32, 631.961, 1290.63, 588.795, and 811.053 millimetres represents the 2015 average annual rainfall across Nigeria. These measurements were obtained from the official records of the Nigerian Meteorological Agency (NIMET), and are utilised in this study as an empirical basis for modelling and comparative analysis.

Table 11: Goodness of Fit Measures for 2015 average annual rainfall across Nigeria

Model	θ	β	LogLikelihood	AIC	BIC
ORFD	154.2308	8.9144	-87.8102	179.6	182.8
FD	131.0946	2.8364	-192.4297	388.9	392.1
WD	411.2692	0.4701	-136.4430	276.9	280.1

The goodness-of-fit results (Table 11) for the 2015 average annual rainfall data indicate that the ORFD model provides a substantially better fit compared to the competing Fréchet (FD) and Weibull (WD) distributions. The ORFD model achieves the highest log-likelihood value (-87.81), reflecting a stronger agreement with the observed data, and produces the lowest AIC (179.6) and BIC (182.8) values among all models evaluated. In contrast, the FD model shows the poorest performance, with the lowest log-likelihood (-192.43) and the highest AIC and BIC values, indicating a weak fit. The WD model performs moderately but still falls short of the ORFD model, as evidenced by its higher AIC (276.9) and BIC (280.1). Generally, the results clearly demonstrate that the ORFD model is the most suitable and efficient model for describing the 2015 average annual rainfall distribution across Nigeria.

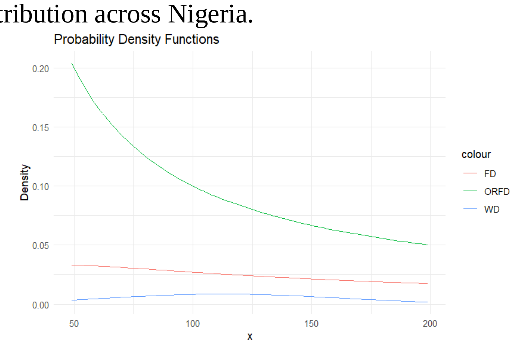

Figure 13: Density Plot for the 2015 average annual rainfall across Nigeria

CONCLUSION

This study introduces the Odd Rayleigh Fréchet Distribution (ORFD) as a novel probability model for analysing monthly total rainfall at the Piracicaba River. Through comparative analysis with existing models, Fréchet Distribution (FD), and Weibull Distribution (WD), the ORFD consistently demonstrated superior performance based on log-likelihood, AIC, and BIC values. The results show that ORFD provides the best fit for the dataset, making it a suitable model for capturing heavy-tailed rainfall distributions. Furthermore, a simulation study was conducted to evaluate the bias and root mean square error (RMSE) of the ORFD's parameter estimates. The results confirm that as the sample size increases, the bias and RMSE decrease, indicating the efficiency and consistency of the proposed model. These findings reinforce the potential of ORFD in hydrological modelling, climate studies, and extreme weather forecasting. The study highlights the practical applicability of ORFD in accurately modelling rainfall distributions, which can aid hydrologists, meteorologists, and policymakers in making informed decisions regarding water resource management and climate adaptation strategies. Future research can explore extending the ORFD model to other hydrological datasets and further refining its applicability in environmental and climate science. Based on the findings of this study, the following recommendations are proposed: adoption of the ORFD model for the Nigerian Meteorological Agency (NIMET)for analysing rainfall extremes and hydrological data; adoption of the ORFD model for hydrological studies, for its superior performance in modelling total monthly rainfall at the Piracicaba River, in Brazil, primarily in the state of São Paulo, the Odd Rayleigh FréchetDistribution (ORFD) should be considered as a suitable probability model for analyzing rainfall extremes and hydrological data; the ORFD can be used by meteorologists and climate scientists for predicting extreme weather events, which is crucial for disaster preparedness, flood risk management, and climate change adaptation strategies; water resource planners and policymakers should incorporate the ORFD model into their hydrological assessments to improve reservoir management, irrigation planning, and drought mitigation efforts; and future studies should explore additional statistical models beyond the ones considered in this study to further validate the robustness of the ORFD in different climatic regions and datasets. By implementing these recommendations, the practical impact of this study can be extended to real-world climate modelling, water resource planning, and extreme weather forecasting, ensuring more accurate and reliable decision-making in environmental and hydrological sciences.

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APPENDIX

Moments and their confidence intervals for different values of θ and β

θ	β	Statistic	Value	CI_Lower	CI_Upper
0.50	1.00	Mean (E[X])	1.037619	0.062209	7.496526
0.50	1.00	Variance	0.127981	-3.870092	59.09904
0.50	1.00	Skewness	0.475407	0.000099	384.47512
0.50	1.00	Kurtosis	3.082353	0.000004	2689.840484
0.50	2.00	Mean (E[X])	1.00316	0.057305	7.534845
0.50	2.00	Variance	0.031289	-4.366496	46.808297
0.50	2.00	Skewness	0.01414	0.00018	385.428284
0.50	2.00	Kurtosis	2.753754	0.000008	2153.605381
0.50	3.00	Mean (E[X])	0.998566	0.04423	6.993238
0.50	3.00	Variance	0.014073	-3.99401	48.233063
0.50	3.00	Skewness	-0.146065	0.000254	483.076412
0.50	3.00	Kurtosis	2.801945	0.000007	3891.805256
1.00	1.00	Mean (E[X])	1.694919	0.027688	3.700528
1.00	1.00	Variance	0.461388	-0.968774	14.446593
1.00	1.00	Skewness	0.555895	0.000025	68.793108
1.00	1.00	Kurtosis	3.167661	0.000001	205.396464
1.00	2.00	Mean (E[X])	1.275004	0.025035	3.781744
1.00	2.00	Variance	0.069284	-1.024192	13.313305
1.00	2.00	Skewness	0.028256	0.000008	54.484619
1.00	2.00	Kurtosis	2.716609	0.000001	113.840195
1.00	3.00	Mean (E[X])	1.170088	0.024848	3.621722
1.00	3.00	Variance	0.026673	-1.027711	11.355869
1.00	3.00	Skewness	-0.156897	0.00001	46.487994
1.00	3.00	Kurtosis	2.77089	0.000002	146.868056
1.50	1.00	Mean (E[X])	2.335852	0.026871	2.367598
1.50	1.00	Variance	1.005578	-0.406352	6.053894
1.50	1.00	Skewness	0.585466	0.000004	17.770289
1.50	1.00	Kurtosis	3.199762	0	43.265329
1.50	2.00	Mean (E[X])	1.491902	0.014117	2.38
1.50	2.00	Variance	0.110082	-0.445173	5.898821
1.50	2.00	Skewness	0.022413	0.000007	11.579735
1.50	2.00	Kurtosis	2.701916	0	42.379823
1.50	3.00	Mean (E[X])	1.29822	0.012407	2.323983
1.50	3.00	Variance	0.038307	-0.422414	5.217454
1.50	3.00	Skewness	-0.177187	0.000005	16.04427
1.50	3.00	Kurtosis	2.769581	0	38.329847
2.00	1.00	Mean (E[X])	2.970962	0.012206	1.834563
2.00	1.00	Variance	1.762305	-0.234764	3.008647

2.00	1.00	Skewness	0.600054	0.000003	6.846291
2.00	1.00	Kurtosis	3.2153	0	11.396583
2.00	2.00	Mean (E[X])	1.678881	0.012573	1.817988
2.00	2.00	Variance	0.15232	-0.269353	3.414174
2.00	2.00	Skewness	0.014088	0.000001	5.4437
2.00	2.00	Kurtosis	2.695428	0	12.593696
2.00	3.00	Mean (E[X])	1.403757	0.010788	1.87635
2.00	3.00	Variance	0.049139	-0.245357	2.822404
2.00	3.00	Skewness	-0.19546	0.000002	5.428959
2.00	3.00	Kurtosis	2.776511	0	16.858919