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Linearization of Chazy-type Third Order Nonlinear Ordinary Differential Equation using the Generalized Sundman Transformation

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ABSTRACT
The paper studies the linearization of a Chazy-type third-order nonlinear ordinary differential equation using the Generalized Sundman Transformation (GST). Nonlinear differential equations are generally difficult to solve analytically, so transforming them into linear equations makes it easier to obtain exact solutions. The work derives the necessary conditions for GST linearization and applies them to the Chazy equation $y''' + 3yy'' + 3(y')^2 = 0$. By constructing suitable transformation functions, the nonlinear equation is converted into the linear third-order equation $U_{TTT} = 0$. The linear equation is then solved, and the solution of the original equation is obtained through back-substitution. Overall, the study demonstrates that the generalized Sundman transformation is an effective method for linearizing certain third-order nonlinear differential equations and obtaining their exact analytical solutions.

Keywords:
Chazy-type;
Differential Equation;
Generalized Sundman Transformation (GST);
Linearization;
Third-Order;
Nonlinear ODE.

INTRODUCTION

A key component of modeling phenomena in physics, biology, and engineering is the solution of nonlinear ordinary differential equations (ODEs). Nonlinear ODEs frequently need either approximation or numerical techniques, whereas linear ODEs can be solved analytically. Because they enable the acquisition of exact solutions, transformations that linearize nonlinear ODEs are consequently extremely valuable.

Suksern & Sookcharoenpinyo, (2023) developed a novel technique for using point transformation to convert nonlinear third-order ordinary differential equations into linear forms. These linear equations are more universal and easier to solve. Their work's primary feature was the use of examples to demonstrate how the derived linearization criteria could be applied to a range of contemporary issues, including second-order ordinary differential equations under the Riccati transformation, third-order ordinary differential equations, and third-order partial differential equations under travelling wave solutions.

If certain essential and sufficient criteria are met, the generalized Sundman transformation, $X = F(t,x)$, $dT = G(t,x)$, $(F_x G \neq 0)$, can convert the most general third-order ordinary differential equation, $x''' = f(t,x,x',x'')$, to $X''' + \alpha X = 0$ (Nakpim & Meleshko, 2010), where $\alpha \neq 0$ is a constant.

The third-order ordinary differential equation linearization problem under the generalized linearizing transformation was examined by Thailert & Suksern, (2014). They outline the requirements that allow the third-order ordinary differential equation to be transformed into the simplest linear equation, as well as the form of the linearizable equations. Additionally, the generalized linearizing transformation's construction was shown. A few examples of linearizable equations were provided to demonstrate their methodology.

The term "Chazy-type third-order nonlinear ordinary differential equation" describes a class of third-order nonlinear ODEs that the French mathematician Jean Chazy (1882–1955) examined when looking into differential equations with unique kinds of singularities in their solutions.

The method for determining the coefficients of the third-order Chazy differential equation with six constant poles in relation to the unknown function is provided (Chichurin, 2017). It is possible to integrate a similar differential equation in symbolic form for such pole values. In order to solve this problem, a computational-algebraic approach was used to generate five third-order non-linear differential equations, which are then reduced to second-order linear inhomogeneous equations with six singular points.

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The point symmetries of the Chazy Equation were shown to transform into nonlocal symmetries for the simplified equation. Additionally, we find that the point symmetries of the Chazy Equation are generalized symmetries for the new equation by building an equivalent third-order differential equation that is connected to the Chazy Equation under a generalized transformation (Paliathanasis et al., 2018). They created a solution for the Chazy Equation, which is provided by a Right Painlevé Series, using singularity analysis and a straightforward coordinate transformation. When they apply the singularity analysis to the new third-order equation, we see that it admits two solutions, one of which is provided by a Left Painlevé Series and the other by a Right Painlevé Series, where the resonances and leading-order behaviors are clearly those of the Chazy Equation.

The unimodular Lie group $SL(2,\mathbb{C})$ has three distinct actions on a two-dimensional space (Clarkson & Olver, 1996). In each instance, the authors demonstrated how an ordinary differential equation admitting $SL(2)$ as a symmetry group may be lowered in order by three. The solution to a linear second order equation can then be recovered from the reduced equation using two quadratures. As an illustration, the Chazy equation, whose general solution can be written as the ratio of two hypergeometric equation solutions was considered. An unexpected transformation between the Lameños and hypergeometric equations results from the reduction approach, which yields an alternate formula in terms of solutions to the Lameň equation. Lastly, they went over the Painleve analysis of the Chazy equation's singularities.

Several significant second order nonlinear ODEs were previously linearized using the generalized Sundman transformation method. The General Modified Second-Order Lane-Emden differential equation was linearized using the GST by Orverem & Nworah, (2025). Additionally, Orverem, (2024) used the generalized Sundman transformation to solve the non-linear Modified Langumir and Van der Pol differential equations.

A third-order nonlinear differential equation of the form

$$y''' = F(x, y, y', y'')$$

is a Chazy-type equation if it falls into a particular class that Chazy identified in his study of equations whose general solutions have movable singularities but no movable critical points (a property connected to the Painlevé property). The categorization of differential equations with single-valued solutions around movable singularities frequently includes these equations.

The generalized Sundman transformation is used in this article to linearize the third order nonlinear Chazy-type ordinary differential equation.

MATERIALS AND METHODS

The Generalized Sundman Transformation

Let a third-order ODE be given by the general third order differential equation:

$$y''' = F(x, y, y', y''). \quad (1)$$

The generalized Sundman transformation (GST) is defined as:

$$T = \int \phi(x, y)\, dx, \qquad U = \psi(x, y), \quad (2)$$

with $\phi(x,y) \neq 0$and $\psi_y \neq 0$. Under this transformation:

$$\frac{d}{dT} = \frac{1}{\phi(x,y)}\frac{d}{dx}. \quad (3)$$

The derivatives transform as:

$$U_T = \frac{\psi_x + \psi_y y'}{\phi},$$

$$U_{TT} = \frac{\left(\psi_{xx} + 2\psi_{xy}y' + \psi_{yy}(y')^2 + \psi_y y''\right)\phi - (\psi_x + \psi_y y')(\phi_x + \phi_y y')}{\phi^3},$$

$$U_{TTT} = \frac{1}{\phi^3}\left[\psi_y y''' + C_1(y'')^2 + C_2 y'' y' + C_3 y'' + C_4 (y')^3 + C_5 (y')^2 + C_6 y' + C_7\right],$$

where the coefficients $C_1, \ldots, C_7$ are:

$$C_1 = \psi_{yy} - (\phi_y/\phi)\psi_y = 0,$$

$$C_2 = 3\psi_{xy} - 3(\phi_x/\phi)\psi_y - 2(\phi_y/\phi)\psi_x = 0,$$

$$C_3 = 3\psi_{xx} - 3(\phi_x/\phi)\psi_x = 0,$$

$$C_4 = \psi_{yyy} - 3(\phi_y/\phi)\psi_{yy} - 3(\phi_{yy}/\phi)\psi_y + 6(\phi_y^2/\phi^2)\psi_y = 0,$$

$$C_5 = 3\psi_{xyy} - 3(\phi_x/\phi)\psi_{yy} - 6(\phi_y/\phi)\psi_{xy} - 3(\phi_{xy}/\phi)\psi_y + 12(\phi_x\phi_y/\phi^2)\psi_y = 0,$$

$$C_6 = 3\psi_{xxy} - 6(\phi_x/\phi)\psi_{xy} - 3(\phi_y/\phi)\psi_{xx} - 3(\phi_{xx}/\phi)\psi_y + 6(\phi_x^2/\phi^2)\psi_y = 0,$$

$$C_7 = \psi_{xxx} - 3(\phi_x/\phi)\psi_{xx} - 3(\phi_{xx}/\phi)\psi_x + 6(\phi_x^2/\phi^2)\psi_x = 0.$$

An alternative formulation can also be achieved with the aid of GST condition. Substituting $\psi_y = C\phi$ into the coefficients:

(i) $C_1 = 0$ automatically.

(ii) The remaining conditions reduce to PDEs for $\phi(x, y)$:

$$C_2 = 3\psi_{xy} - 3(\phi_x/\phi)C\phi - 2(\phi_y/\phi)\psi_x = 0,$$

$$C_3 = 3\psi_{xx} - 3(\phi_x/\phi)\psi_x = 0,$$

$$C_4 = \psi_{yyy} - 3(\phi_y/\phi)\psi_{yy} - 3(\phi_{yy}/\phi)C\phi + 6(\phi_y^2/\phi^2)C\phi = 0,$$

etc.

(iii) This overdetermined system can be solved for $\phi(x,y)$and then $\psi(x,y) = C\int \phi dy + f(x)$.

At this juncture, the necessary and sufficient condition of GST linearizability can be obtained.

1. There exists a nonzero smooth function $\phi(x,y)$and a constant $C \neq 0$ such that $\psi_y = C\phi$.
2. The coefficients $C_2, C_3, \ldots, C_7$vanish identically: $C_i(x, y, \phi, \phi_x, \phi_y, \phi_{xx}, \ldots) = 0, i = 2, \ldots, 7$
3. The original equation can then be written in a form where all nonlinear derivative terms are removed under the GST.

These conditions must be satisfied for the third order ordinary differential equation to be GST-linearizable.

The following is the generalized Sundman transformation linearization algorithm:

1. Check the GST condition $\psi_y = C\phi$.
2. Compute coefficients $C_1, \ldots, C_7$— if they vanish, the equation is GST-linearizable.
3. Choose a transformation $U = \psi(x,y)$and $T = \int \phi dx$.
4. Substitute to reduce to a linear third-order ODE (or separable form).
5. Solve and back-substitute to obtain $y(x)$.

The relationship between the original dependent variable $y(x)$, the GST reparametrized independent variable $T(x)$, and the transformed dependent variable $U(T)$ is shown in this GST transformation diagram.

$y(x)$ —> $T(x)$ $\left(T = \int \varphi(x,y)\, dx\right)$

└——> $U(T)$ $\left(U = \psi(x,y)\right)$

▼

$Linear\ ODE: U_{TTT} = 0$

The arrows denote transformations.

GST simultaneously reparametrizes x and modifies y to linearize the ODE.

GST Linearization of Chazy-Type Third order nonlinear ordinary Differential Equation

Consider the third-order nonlinear ODE kwon as the Chazy differential equation:

$$y''' + 3yy'' + 3(y')^2 = 0. \tag{4}$$

We want to:

1. Determine if equation (4) is GST-linearizable.
2. Find the transformation $T = \int \phi(x,y)dx$, $U = \psi(x,y)$.
3. Solve the linearized equation.

The Generalized Sundman Transformation is:

$$T = \int \phi(x,y)dx, \quad U = \psi(x,y) \tag{5}$$

with

$$U_T = \frac{\psi_x + \psi_y y'}{\phi},$$

$$U_{TTT} = \frac{1}{\phi^3}\left[\psi_y y''' + C_1 y''^2 + C_2 y'' y' + \cdots + C_7\right].$$

Necessary condition is now

$$\psi_y = C\phi.$$

For simplicity, take $C = 1$. Then:

$$\psi_y = \phi. \tag{6}$$

We need to cancel the nonlinear $y''$and $(y')^2$ terms of (4) in U_{TTT}. Recall that

$$C_1 = \psi_{yy} - \left(\frac{\phi_y}{\phi}\right)\psi_y.$$

Substitute $\psi_y = \varphi$, so that

$$C_1 = \phi_y - (\phi_y/\phi)\phi = 0. \tag{7}$$

This condition is satisfied automatically.

Next, for the coefficient of $y''y'$:

$$C_2 = 3\psi_{xy} - 3\left(\frac{\phi_x}{\phi}\right)\psi_y - 2\left(\frac{\phi_y}{\phi}\right)\psi_x$$
$$= 3\psi_{xy} - 3\left(\frac{\phi_x}{\phi}\right)\phi - 2\left(\frac{\phi_y}{\phi}\right)\psi_x.$$

We assume φ depends on y only: $\varphi = \varphi(y)$. Then $\varphi_x = 0, \ \varphi_y \neq 0$.

Then:

$$C_2 = 3\psi_{xy} - 2(\phi_y/\phi)\psi_x = 0. \tag{8}$$

We can solve (8) by choosing $\psi_x = 0$, this implies that $C_2 = 0$ automatically.

Next, coefficient of $(y')^2$ term is:

$$C_4 = \psi_{yyy} - 3\left(\frac{\varphi_y}{\varphi}\right)\psi_{yy} - 3\left(\frac{\varphi_{yy}}{\varphi}\right)\psi_y + 6\left(\frac{\varphi_y^2}{\varphi^2}\right)\psi_y.$$

Substitute $\psi_y = \varphi \Rightarrow \psi_{\{yy\}} = \varphi_y, \ \psi_{\{yyy\}} = \varphi_{\{yy\}}$ one has that:

$$C_4 = \varphi_{yy} - 3\left(\frac{\varphi_y}{\varphi}\right)\varphi_y - 3\left(\frac{\varphi_{yy}}{\varphi}\right)\varphi + 6\left(\frac{\varphi_y^2}{\varphi^2}\right)\varphi$$
$$= \varphi_{yy} - \frac{3\varphi_y^2}{\varphi} - 3\varphi_{yy} + \frac{6\varphi_y^2}{\varphi}$$
$$= -2\varphi_{yy} + \frac{3\varphi_y^2}{\varphi}.$$

Setting $C_4 = 0$, we see that:

$$-2\varphi_{yy} + \frac{3\varphi_y^2}{\varphi} = 0 \implies 2\varphi_{yy} = \frac{3\varphi_y^2}{\varphi} \implies \frac{\varphi_{yy}}{\varphi_y^2} = \frac{3}{2\varphi}.$$

This is a first-order ordinary differential equation for φ_y.

Let $p = \varphi_y$:

$$\frac{dp}{dy} = \frac{3}{2}\frac{p^2}{\varphi}.$$

Also, $p = d\varphi/dy$. Then:

$$\frac{d^2\varphi}{dy^2} = \frac{3}{2}\frac{(d\varphi/dy)^2}{\varphi}. \tag{9}$$

Equation (9) is nonlinear ODE for $\varphi(y)$.

We now solve equation (9) for $\varphi(y)$. This is a standard Riccati-type ordinary differential equation. Solve by substitution to have:

$$p = \varphi_y, \qquad p' = \varphi_{yy} = \frac{\frac{3}{2}p^2}{\varphi},$$

and then,

$$\frac{dp}{dy} = \frac{3}{2}\frac{p^2}{\varphi}.$$

But $\frac{dp}{dy} = \left(\frac{dp}{d\varphi}\right)\left(\frac{d\varphi}{dy}\right) = p\frac{dp}{d\varphi}$. Therefore,

$$p\frac{dp}{d\varphi} = \frac{3}{2}\frac{p^2}{\varphi} \implies \frac{dp}{d\varphi} = \frac{3}{2}\frac{1}{\varphi} \implies dp = \frac{3}{2}\frac{d\varphi}{\varphi} \implies p = \frac{3}{2}\ln\varphi + k.$$

In other words:

$$dp = \frac{3}{2}\frac{d\varphi}{\varphi} \implies p = \frac{3}{2}\ln\varphi + c_1.$$

Then:

$$\varphi_y = p = \frac{3}{2}\ln\varphi + c_1. \quad (10)$$

Equation (10) is still a nonlinear ODE, but can be solved implicitly. For simplicity, we can take $c_1 = 0$, then:

$$\frac{d\varphi}{dy} = \frac{3}{2}\ln\varphi.$$

Separate to have

$$\frac{d\varphi}{\ln\varphi} = \frac{3}{2}dy,$$

and integrate to have:

$$\mathrm{Li}(\varphi) = \frac{3}{2}y + C,$$

where $Li(\varphi)$ is the logarithmic integral. This gives $\varphi(y)$ implicitly.

Now, we solve for $\psi(x,y)$ as:

$$\psi_y = \varphi(y) \implies \psi = \int\varphi(y)dy + f(x),$$

where $f(x)$ can be chosen 0, $\psi(x,y) = \int\varphi(y)\,dy$ (implicit form).

Under the generalized Sundman transformation (GST),

$$T = \int\varphi(y)dx, \qquad U = \psi(x,y),$$

then

$$U_{TTT} = 0. \quad (11)$$

Integrate (11) to have:

$$U(T) = c_0 + c_1T + c_2T^2.$$

Finally, carry out back substitution to get:

$$\psi(x,y) = U(T) \implies \int\varphi(y)dy = c_0 + c_1\int\varphi(y)dx + c_2\left(\int\varphi(y)dx\right)^2.$$

If we choose $\varphi(y) = e^y$ a fully explicit solution will be produced. We now have that:

$$T = \int\phi(y)dx = \int e^y dx, \quad (12)$$

and

$$\psi(x,y) = \int\phi(y)dy = \int e^y dy = e^y.$$

So, the GST is:

$$T = \int e^y dx,\ U = e^y. \quad (13)$$

From the first equation of (13), $T = \int e^y dx \implies dT = e^y dx \implies dx = e^{-y}dT$.

Recall that under GST

$$U_{TTT} = 0,$$

so that in transformed coordinates

$$\frac{d^3U}{dT^3} = 0. \quad (14)$$

Integrate (14) three times, do back substitution and solve explicitly to have:

$$\frac{dT}{dx} = c_2T^2 + c_1T + c_0. \quad (15)$$

This is a standard Riccati equation for $T(x)$ quadratic in T.

Case $c_2 \neq 0$

Integrate (15) via partial fractions

$$dx = \frac{dT}{c_2T^2 + c_1T + c_0},$$

solve the quadratic in denominator

$$\Delta = c_1^2 - 4c_0c_2.$$

Then:

$$x + x_0 = \frac{1}{\sqrt{\Delta}}\ln\left|\frac{2c_2T + c_1 - \sqrt{\Delta}}{2c_2T + c_1 + \sqrt{\Delta}}\right|.$$

Solve for T to have

$$T = \frac{\sqrt{\Delta}}{2c_2}\frac{1 + Ke^{\sqrt{\Delta}(x+x_0)}}{1 - Ke^{\sqrt{\Delta}(x+x_0)}} - \frac{c_1}{2c_2}, \quad (16)$$

where K is an integration constant.

Thus, using the back substitution we have

$$e^y = U = c_0 + c_1T + c_2T^2.$$

So, finally

$$y(x) = \ln\,[c_0 + c_1T(x) + c_2T(x)^2], \quad (17)$$

where $T(x)$ is given explicitly in (16).

Taking $c_2 = 0,\ c_1 = 1,\ c_0 = 0$, then equation (15) becomes:

$$\frac{dT}{dx} = T \implies \frac{dT}{T} = dx \implies T = T_0e^x.$$

Then:

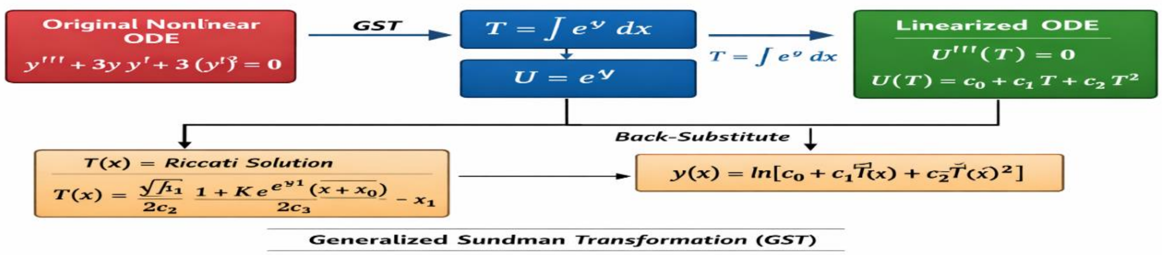
$$y = \ln U = \ln(T) = \ln(T_0e^x) = x + \ln T_0.$$

Fully explicit and simple solution is now:

$$y(x) = x + \text{const}.$$

This is a special solution satisfying the original ordinary differential equation. For the general solution, we use the quadratic formula for $T(x)$ as in equation (15). Note that the fully explicit solution in closed form is the solution in equation (17).

Below is the GST diagram for the Chazy-type third order ordinary differential equation.



RESULTS AND DISCUSSION

The findings demonstrate that the Generalized Sundman Transformation (GST) may effectively linearize the nonlinear Chazy-type third-order differential equation $y''' + 3yy'' + 3(y')^2 = 0$. The nonlinear equation is transformed into the simple linear equation $\frac{d^3U}{dT^3} = 0$, which is readily solved by direct integration, by inserting the transformations $T = \int \varphi(y)dx$ and $U = \psi(x, y)$. By ensuring that the coefficients connected to nonlinear derivative terms disappear, the study determines the prerequisites for this linearization. This results in a set of equations for the transformation functions $\phi(x, y)$ and $\psi(x, y)$, which guarantees the elimination of the nonlinear terms in the original equation following the transformation.

By making an appropriate decision $\phi(y) = e^y$, the paper constructs an explicit GST transformation and derives the original equation's analytical solution. The general solution for $y(x)$ is obtained by back substitution after the converted linear equation is integrated three times. Additionally, a simple special solution $y(x) = x + constant$ is found. These findings show that the GST offers a methodical and efficient way to reduce some nonlinear third-order differential equations to linear form, enabling the derivation of explicit analytical solutions and indicating that the method may be applied to other higher-order nonlinear differential equations.

CONCLUSION

The paper shows that a Chazy-type third-order nonlinear differential equation can be successfully linearized using the Generalized Sundman Transformation. By applying this transformation, the nonlinear equation $y''' + 3yy'' + 3(y')^2 = 0$, originally studied by Jean Chazy, is converted into the simpler linear equation $U_{TTT} = 0$.

The study demonstrates that GST provides a systematic method for transforming certain nonlinear third-order differential equations into solvable linear forms, making it possible to obtain explicit analytical solutions. The work also suggests that this approach can be extended to other higher-order nonlinear differential equations in applied mathematics.

Conflicts of Interest

Regarding the publication of this study, the authors state that there is no conflict of interest. The study was carried out independently, and neither the results nor their interpretation were impacted by any institutional, financial, or personal ties.

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