Journal of Basics and Applied Sciences Research (JOBASR)
ISSN (print): 3026-9091, ISSN (online): 1597-9962
Volume 4(2) March 2026
DOI: https://dx.doi.org/10.4314/jobasr.v4i2.8

Exponentiated Student-t Distribution: Properties and modelling of financial data using Asymmetric GARCH models.

Okereke, Catherine Uloaku[1*], Emmanuel John Ekpenyong[2] & Usoro, Anthony Effiong[3]
[1]Department Of Statistics, Michael Okpara University of Agriculture Umudike, Nigeria
[2]Department of Mathematics /Statistics, Federal University of Technology, Ikot- Abasi, Nigeria
[3]Department of Statistics, Akwa Ibom State University, Mkpat Enin, Akwa Ibom, Nigeria
[*]Corresponding Author Email: okereke.catherine@mouau.edu.ng

ABSTRACT
This study proposes a novel error innovation, the Exponentiated Student-t Distribution (ESTD), to enhance volatility modeling within asymmetric GARCH frameworks. The performance of the proposed innovation is evaluated against conventional distributions—Student-t and Generalized Error Distribution (GED)—using both simulated data and empirical returns from the S&P 500 index. Two asymmetric GARCH models, EGARCH (1,1) and GJR-GARCH (1,1), are employed to capture volatility dynamics, including leverage effects. Model evaluation considers both in-sample fit (log-likelihood, AIC, BIC) and out-of-sample forecasting accuracy (MSE, RMSE, MAE). Results consistently indicate that models incorporating ESTD outperform the benchmarks across both model specifications, yielding higher log-likelihood values, lower information criteria, and reduced forecast errors. The persistence measures under the ESTD specification remain below unity and are comparable to those obtained under GED and Student-t assumptions The GJR-GARCH (1,1) model with ESTD exhibits the strongest overall performance, effectively capturing asymmetric volatility responses and extreme market fluctuations. Economically, the ESTD innovation improves the representation of fat tails and extreme shocks, providing more reliable volatility forecasts for financial risk management, derivative pricing, and portfolio allocation. The findings demonstrate the practical advantages of the ESTD distribution over conventional innovations, highlighting its contribution to the literature on volatility modeling. This study therefore contributes to the growing literature on flexible innovation distributions in GARCH-type models by introducing an exponentiated heavy-tailed specification that improves the modeling of extreme financial returns without substantially increasing estimation complexity.

Keywords:
Volatility Modelling,
Asymmetric GARCH,
Student -t,
Exponentiation,
Error distribution,
S&P 500.

INTRODUCTION

Accurate and reliable predictions of financial time series remain a major challenge due to some of its stylized properties such as excess kurtosis, skewness, heavy tail, volatility clustering and leverage effects. Despite efforts made at providing a reliable estimate of the volatility of financial asset, the performance of this volatility models continues to be a subject of active research and debate, with various studies finding different models to be superior depending on the specific data, market conditions, and evaluation metrics used**.** The performance of these volatility models do not only depend on the robustness of the volatility model but also on the innovation density used [1] [2][3].

The estimation of the parameters of these volatility models were usually estimated using an innovation density which implies that there is a distributional assumption associated with these models. The first of the innovation density proposed for the estimation of the parameters of this heteroscedasticity model is the normal or Gaussian distribution proposed by [4]. The Gaussian distribution assumes that the innovation follows a normal distribution but due to the nature of financial series which often in reality departs from normality, this led to the development of other distributions that are non-normal. Some of these distributions include the student-t distribution proposed by Bollerslev [5] and generalized error distribution by Nelson [6] and among others.

The student-t distribution assumes that the error innovation follows a student -t distribution while generalized error distribution made assumption that the distribution of the innovation can be approximated by generalized distribution. These two distributions have been widely used in several studies and it has been established that these distributions outperformed normal distribution within GARCH family of model frameworks [7] [8][9]. These distributions despite their asymmetric nature and ability to provide a better forecasting accuracy over the normal distribution, they both do not have skewness parameter. There have been also continued researches aimed at generating new classes of distribution in volatility modeling to improve the performance of the distribution in view of the changing nature of financial data [10][11]. These studies employed the use of the exponentiation method proposed by [12], where one shape parameter is introduced into their parent distribution to develop an exponentiated generalized skewed student t distribution (EGSSTD) [13], exponentiated skewed student t error distribution, the exponentiated half logistic skew-t distribution by [14]. These distributions to some extents have been able to capture the volatility in financial series. Despite these advancements, the introduction of additional shape parameters in the exponentiated skewed distributions may lead to over-parameterization. It is therefore very imperative to leverage on the method of generating flexible distribution to generate a new non-Gaussian distribution for the estimation of the volatility of financial time series. This research aims to address this issue of overparameterization by applying the method of exponentiation directly to non-Gaussian innovation densities, specifically the student's t-distribution to provide a more parsimonious yet flexible modeling framework for financial volatility, potentially offering more accurate and reliable estimates of asset volatility. Although flexible innovation distributions in GARCH-type specifications have been extensively investigated, the role of exponentiated heavy-tailed transformations on the baseline distribution remains unexplored. This study fills this gap by introducing an Exponentiated Student-t distribution that enhances tail adaptability without substantially increasing model complexity. By embedding this distribution within EGARCH and GJR-GARCH frameworks and conducting a comprehensive empirical evaluation using S&P 500 returns, this study provides new evidence that exponentiated heavy-tailed innovations can significantly improve volatility and extreme-return modeling.

MATERIALS AND METHODS

Exponentiated Student t –Distribution (ESTD)

If X is a continuous random variable with probability density function, then, $F(x)$ is the Cumulative Density Function (CDF) of the proposed exponentiated distribution $G(x)$, which can be expressed as

$$G(x) = F(x)c \quad , C > 0 \tag{1}$$

where F(x) is the cumulative density function of the parent distribution (student -t distribution) and is given by:

$$F(x)=\frac{1}{2}+\frac{x.B\left(\frac{v}{2},\frac{1}{2}\right)}{2\sqrt{v}B\left(\frac{1}{2},\frac{v}{2}\right)}\left(1+\frac{x^2}{v}\right)^{-\frac{v+1}{2}},-\infty<x<\infty \tag{2}$$

where, v is the degrees of freedom and B(a,b) = $\frac{\Gamma a\Gamma b}{\Gamma a+b}$ is a beta function

The corresponding Probability Density Function (pdf) of the exponentiated distribution is given as:

$$g(x)=c\left[F(x)\right]^{c-1} f(x), c>0 \tag{3}$$

Where f(x) is the pdf of the parent distribution and is given by

$$f(x)=\frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{v\pi}\Gamma\left(\frac{v}{2}\right)}\left(1+\frac{x^2}{v}\right)^{-\frac{v+1}{2}},-\infty<x<\infty \tag{4}$$

Substituting equation (2) into (1) gives

$$G(x)=\left[\frac{1}{2}+\frac{x.B\left(\frac{v}{2},\frac{1}{2}\right)}{2\sqrt{v}B\left(\frac{1}{2},\frac{v}{2}\right)}\left(1+\frac{x^2}{v}\right)^{-\frac{v+1}{2}}\right]^c \tag{5}$$

And substituting equation (4) into (3) gives

$$\text{g(x)=c}\left[\frac{1}{2}+\frac{x\beta\left(\frac{v}{2},\frac{1}{2}\right)}{2\sqrt{v}.\beta\left(\frac{1}{2},\frac{v}{2}\right)}\left(1+\frac{x^2}{v}\right)^{-\frac{v+1}{2}}\right]^{c-1}.\frac{\Gamma\left(\frac{v+1}{2}\right)}{\Gamma(\frac{v}{2})\sqrt{v\Pi}}\left(1+\frac{x^2}{v}\right)^{-\frac{v+1}{2}} \tag{6}$$

Where G(x) and g (x) are the cumulative density function and the probability density function of the new exponentiated student-t distribution respectively. The parameter c is the shape parameter.

The pdf of the Exponentiated Student -t distribution

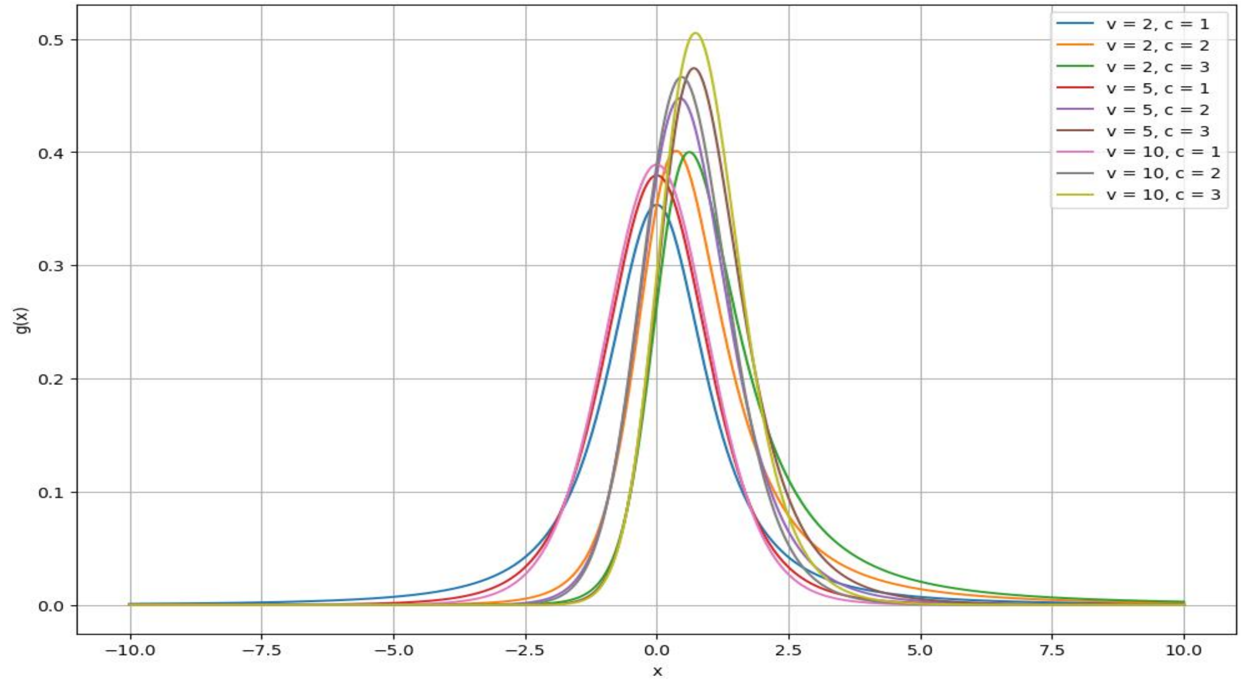

Figure 1: The graph of the PDF of the Exponentiated Student -t distribution.

The graph shows that as the parameter c increases, the distribution tends to a normal distribution and further enhances the peakedness, effectively controlling the spread and the kurtosis of the distribution.

The CDF of the Exponentiated Student -t distribution

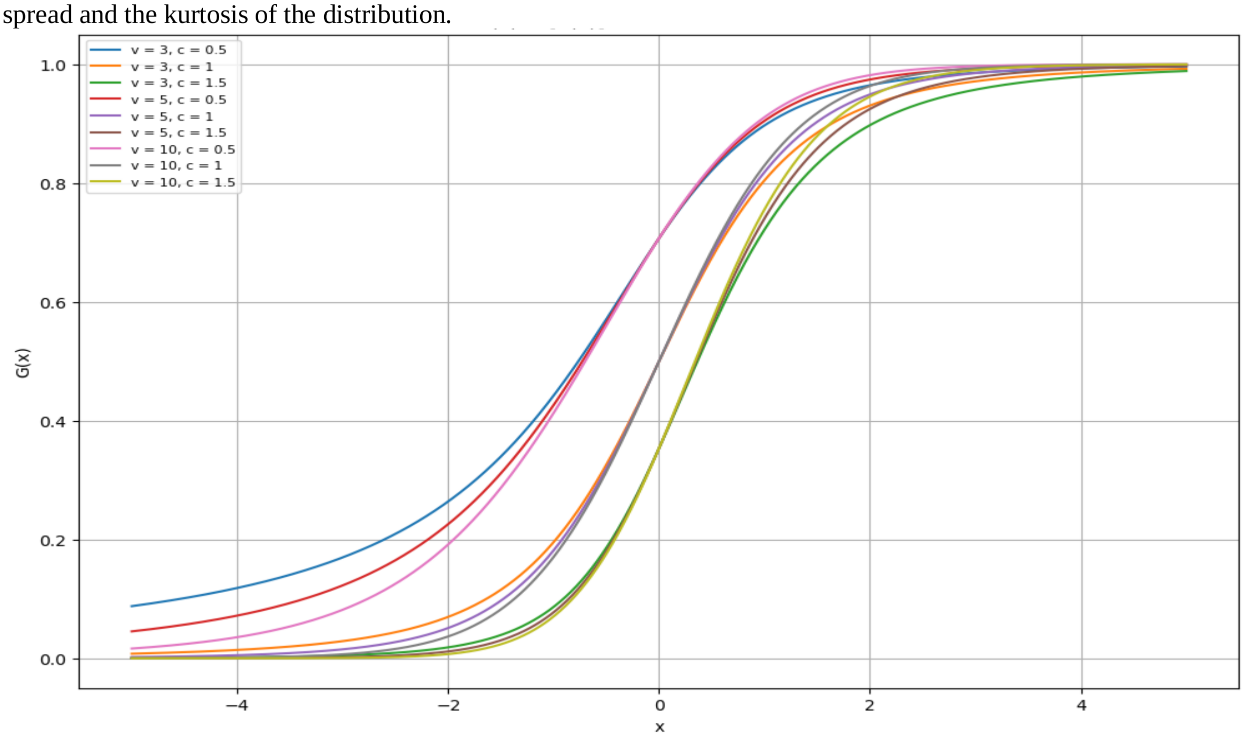

Figure 2: The graph of CDF of the Exponentiated Student -t distribution

This CDF plot for the Exponentiated Student t-distribution demonstrates that higher values of c make the CDF steeper, indicating that the probability mass is more concentrated, resulting in a more sharply peaked distribution around its central tendency**.** These parameters together control the location and dispersion of the Exponentiated Student t-distribution

Statistical properties of the Exponentiated Student-t distribution

Hazard Function of the proposed Exponentiated Student-t Distribution

The hazard function is used in volatility modeling, within the framework of survival analysis**,** to model the time until a specific event occurs, such as a financial asset's price changing. The hazard function of a random variable x with density function g(x) and a cumulative density function G(x) is given by:

$$h(x) = \frac{g(x)}{1-G(x)}$$

$$h(x) = \frac{c.F(x)^{c-1}.f(x)}{1-F(x)^{c}} \qquad (7)$$

The hazard function of the proposed ESTD is given as:

$$h_g(x) = \frac{c\left[\frac{1}{2}+\frac{x.B\left(\frac{v}{2},\frac{1}{2}\right)}{2\sqrt{v}.B\left(\frac{1}{2},\frac{v}{2}\right)}\left(1+\frac{x^2}{v}\right)^{-\frac{v+1}{2}}\right].\frac{\Gamma\left(\frac{v+1}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\sqrt{v\pi}}\left(1+\frac{x^2}{v}\right)^{-\frac{v+1}{2}}}{1-\left[\frac{1}{2}+\frac{x.B\left(\frac{v}{2},\frac{1}{2}\right)}{2\sqrt{v}B\left(\frac{1}{2},\frac{v}{2}\right)}\left(1+\frac{x^2}{v}\right)^{-\frac{v+1}{2}}\right]} \qquad (8)$$

Hazard function of the Exponentiated Student -t distribution

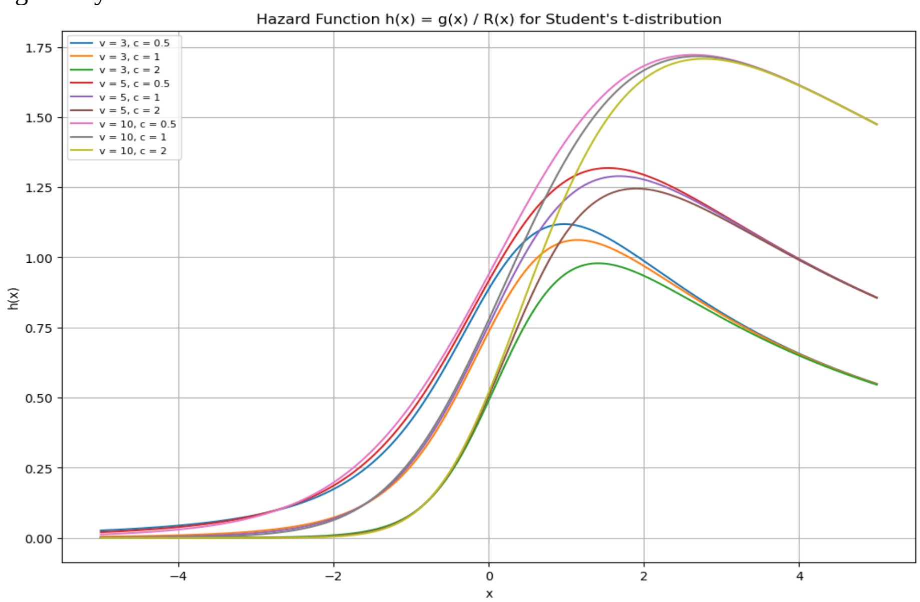

Figure 3: The graph of the hazard function of the Exponentiated Student -t distribution

The graph reveals that the parameter v shifts the peak risk to later x values and generally increases the peak magnitude while parameter c sharpens the peak risk, making the period of high risk more concentrated and increasing the maximum instantaneous risk.

Reliability Function of the Exponentiated Student -t distribution (ESTD)

The term "reliability function" in the context of volatility modeling refers to the **survival function**, which is used to quantify the probability that a financial system will operate without "failure" over a specified period of time.

The reliability function of any probability density function is given as

$$R(x) = P[X > x] = 1 - F(x)$$

$$R(x) = 1-\left[\frac{1}{2}+\frac{x.B\left(\frac{v}{2},\frac{1}{2}\right)}{2\sqrt{v}B\left(\frac{1}{2},\frac{v}{2}\right)}\left(1+\frac{x^2}{v}\right)^{-\frac{v+1}{2}}\right]^{c} \qquad \textbf{(9)}$$

Reliability function of the exponentiated student -t distribution

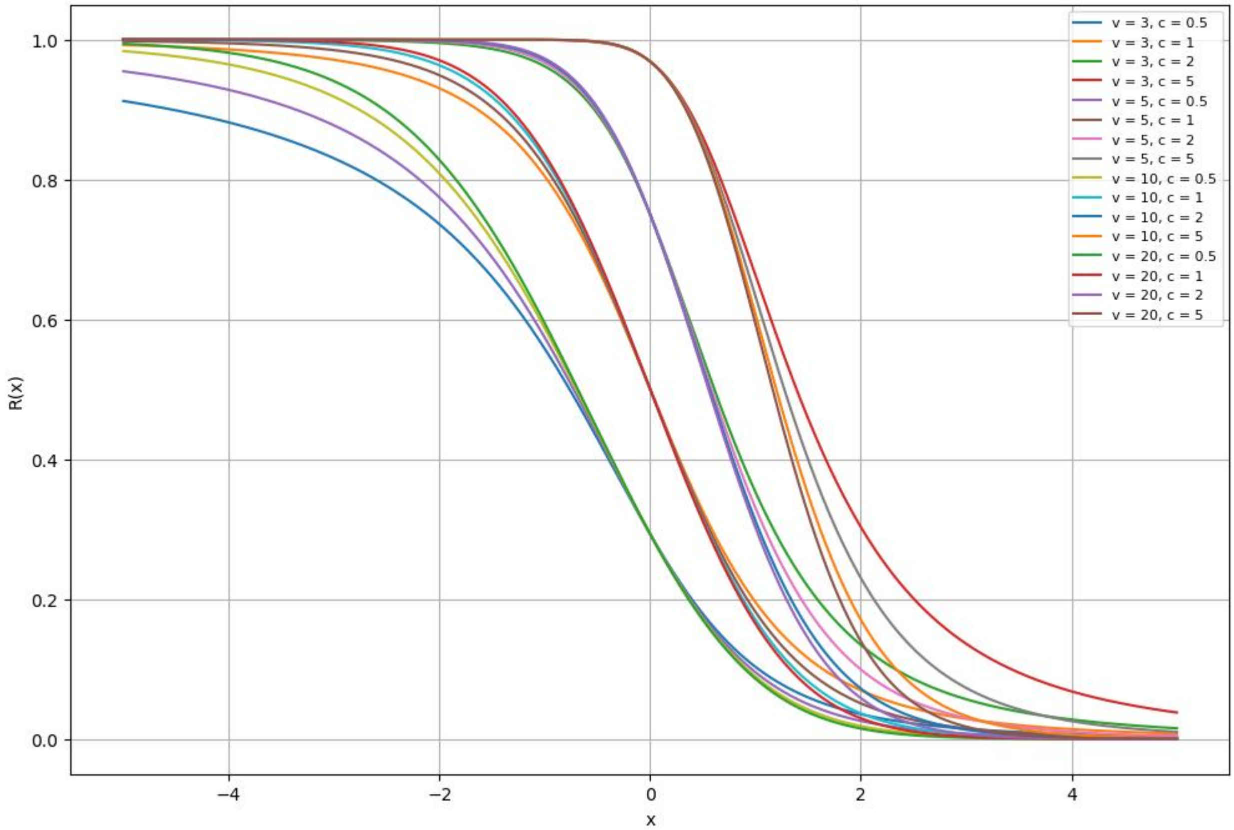

Figure 4: The graph of the reliability function of the exponentiated student – t distribution

The graph of the reliability function shows that higher c values lead to a more abrupt decline in reliability, suggesting a more concentrated "failure time" or a sharper transition from high to low survival probability.

The r^{th} moment of the exponentiated student-t distribution (ESTD)

Moments play a crucial role in volatility modeling, acting as the foundation for measuring, estimating, and forecasting asset price fluctuations.

Let X be a random variable that follows an exponentiated Student- t distribution (ESTD), the r^{th} moment is defined as:

$$E\left[X^r\right] = \int_{-\infty}^{\infty} x^r g(x)dx = c\int_{-\infty}^{\infty} x^r \left[F(x)\right]^{c-1} g(x)dx \qquad (10)$$

Let $u = F(x) \Rightarrow x = F^{-1}(u)$

$$\frac{dx}{du} = \frac{1}{F(x)} = \frac{1}{F\left[F^{-1}(u)\right]}$$

After change of variable, the r^{th} moment becomes

$$E\left[X^r\right] = c\int_0^1 \left[F^{-1}(u)\right]^r (u)^{c-1} \frac{1}{F\left[F^{-1}(u)\right]du} \qquad (11)$$

$$E\left[X^r\right] = c\int_0^1 \left[F^{-1}(u)\right]^r (u)^{c-1} du$$

For the inverse CDF, $F^{-1}(u)$ of the student's t distribution with v degree of freedom

$$F^{-1}(u) = \operatorname{sgn}(u-0.5).\sqrt{\frac{v(1-u)}{u}}$$

Where sgn (u – 0.5) is the sign function. The r^{th} moment

$$E\left[X^r\right] = c\int_0^{0.5} \left(-\sqrt{\frac{v(1-u)}{u}}\right)^r (u)^{c-1} du + c\int_{0.5}^1 \left(-\sqrt{\frac{v(1-u)}{u}}\right)^r (u)^{c-1} du \qquad (12)$$

$$E\left[X^r\right] = c\int_0^{0.5} \left(-\sqrt{\frac{v(1-u)}{u}}\right)^{\frac{r}{2}} (u)^{c-1} du + c\int_{0.5}^1 \left(-\sqrt{\frac{v(1-u)}{u}}\right)^{\frac{r}{2}} (u)^{c-1} du$$

$$E\left[X^r\right] = c(-1)^r v^{r/2} \int_0^{0.5} (1-u)^{\frac{r}{2}} (u)^{c-1-\frac{r}{2}} du + cv^{\frac{r}{2}} \int_{0.5}^1 (1-u)^{\frac{r}{2}} (u)^{c-1-\frac{r}{2}} du \qquad (13)$$

Simplifying using the Beta Function

$$\int_0^1 t^{a-1}(1-t)^{b-1}dt = \mathrm{B}(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

$a,b>0$

$$E\left[X^r\right] = \int_0^{0.5}(1-u)^{r/2}u^{c-1-r/u}du = B_{0.5}(c-\frac{r}{2},\frac{r}{2}+1)$$

Incomplete beta function

And

$$E\left[X^r\right] = \int_{0.5}^{1}(1-u)^{r/2}u^{c-1-r/u}du = B_{0.5}(c-\frac{r}{2},\frac{r}{2}+1)$$

Incomplete beta function

For even r, the term becomes 2, resulting in

$$E(X^r) = 2cv^{\frac{r}{2}}\beta_{0.5}(c-\frac{r}{2},\frac{r}{2}+1) \tag{14}$$

The mean,

$$E(X) = c\int_{-\infty}^{\infty} x\left[F(x)^{c-1}\right]f(x)dx \tag{15}$$

Note that the student -t distribution is symmetric around zero, the mean and the variance will be zero

The expected value $E(X)$ will be zero because of an odd function over symmetric interval is zero

Variance $\left(\sigma^2\right)$ of the exponentiated student -t distribution

$$E(X^2) = c\int_{-\infty}^{\infty} x^2\left[F(x)\right]^{c-1} f(x)dx$$

Using the result obtained for even moments

$$E(X^2) = 2cv\beta_{0.5}(c-\frac{2}{2},\frac{2}{2}+1) = 2cv\beta_{0.5}(c-1,2)$$

$$\mathrm{var}(X) = E(X^2) - \left[E(X)\right]^2$$

Hence,

$$\mathrm{var}(x) = 2cv\beta(c-1,2) \tag{16}$$

The Moment Generating Function (MGF) of the Exponentiated Student-Distribution

The MGF helps model the distribution of underlying asset returns, which is crucial for determining implied volatility.

Generally, the MGF of a random variable X is defined as:

$$M_X(t) = E(e^{tX}) \tag{17}$$

The moment generating function of the proposed exponentiated distribution can be expressed as:

$$M(x) = \int_{-\infty}^{\infty}\exp(tx)g(x)dx$$

$$M_t = c\int_{-\infty}^{\infty} e^{tx}\left[F(x)\right]^{c-1} f(x)dx \tag{18}$$

$$M_t = c\int_{-\infty}^{0} e^{tx}\left[\frac{1}{2}\mathrm{I}\frac{v}{x^2+v}\left(\frac{v}{2},\frac{1}{2}\right)\right]^{c-1}\frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{v\pi}\Gamma(\frac{v}{2})}\left(1+\frac{x^2}{v}\right)^{-\frac{v+1}{2}}dx + c\int_{0}^{\infty} e^{tx}\left[1-\frac{1}{2}\mathrm{I}\frac{v}{x^2+v}\left(\frac{v}{2},\frac{1}{2}\right)\right]^{c-1}\frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{v\pi}\Gamma(\frac{v}{2})}\left(1+\frac{x^2}{v}\right)^{\frac{-v+1}{2}} \tag{19}$$

Note for the CDF, for $x>0$

$$F(x) = 1-\frac{1}{2}I_{\frac{v}{x^2+v}}\left(\frac{v}{2},\frac{1}{2}\right)$$

Where $I_z(a,b)$ is the regularized incomplete beta function.

For $x<0$, we have $F(x) = 1-\frac{1}{2}I_{\frac{v}{x^2+v}}\left(\frac{v}{2},\frac{1}{2}\right)$

Using the relationship between the regularized incomplete beta function and beta function

$$I_z(a,b) = \frac{B(z;a,b)}{B(a,b)}$$

$$m_t = c\int_{-\infty}^{\infty} e^{tx}\left[\frac{1}{2}.\frac{\beta(\frac{v}{x^2+v};\frac{v}{2},\frac{1}{2})}{\beta(a,b)}\right]^{c-1}\frac{\Gamma(\frac{v+1}{2})}{\sqrt{v\pi}\Gamma(\frac{v}{2})}\left(1+\frac{x^2}{v}\right)^{-\frac{v+1}{2}}dx \tag{19}$$

The integral above involves the product of an exponential function and the pdf of student -t distribution. Due to the complexity of the beta function inside the integral, a closed – form evaluation of the MGF is not feasible. This can only be done using numerical solution.

ESTIMATION.

Maximum Likelihood Estimation of the Exponentiated Student -t distribution

For a sample X_1, X_2, …, Xn

the likelihood function L is given as;

$$L(v,c)=\prod_{i=1}^{n} c\left[F(x_i)\right]^{c-1} f(x_i) \tag{20}$$

Taking the natural logarithm;

$$\ln L(v,c)=\sum_{i=1}^{n}\left[\ln c+(c-1)\ln F(x_i)+\ln f(x_i)\right] \tag{21}$$

Substituting F(x) and f(x) in (21) yields

$$\ln L(v,c)=\sum_{i=1}^{n}\left[\begin{array}{l}\ln c+(c-1)\ln(1-\frac{1}{2}\mathrm{I}\frac{v}{v+x_i^2}(\frac{v}{2},\frac{1}{2}) \\ +\ln\frac{\Gamma(\frac{v+1}{2})}{\sqrt{v\pi}\,\Gamma(\frac{v}{2})}-\frac{v+1}{2}\ln(1+\frac{x_i^2}{v})\end{array}\right] \tag{22}$$

Expanding and simplifying (22) results in

$$\ln L(v,c)=\sum_{i=1}^{n}\left[n\ln c+(c-1)\sum_{i=1}^{n}\ln(1-\frac{1}{2}l\frac{v}{v+x_i^2}(\frac{v}{2},\frac{1}{2})\right]+ \tag{23}$$
$$n\ln\Gamma\left(\frac{v+1}{2}\right)-n\ln\Gamma\left(\frac{v}{2}\right)-\frac{n}{2}\ln(v\pi)-\frac{v+1}{2}\sum_{i=1}^{n}\ln\left(1+\frac{x_i^2}{v}\right)$$

For the Exponentiated Student t Distribution, the log-likelihood is given in equation (23)

$$L(v,c)=\sum_{i=1}^{n}\left[n\ln c+(c-1)\sum_{i=1}^{n}\ln(1-\frac{1}{2}l\frac{v}{v+x_i^2}(\frac{v}{2},\frac{1}{2})\right]+$$
$$n\ln\Gamma\left(\frac{v+1}{2}\right)-n\ln\Gamma\left(\frac{v}{2}\right)-\frac{n}{2}\ln(v\pi)-\frac{v+1}{2}\sum_{i=1}^{n}\ln\left(1+\frac{x_i^2}{v}\right)$$

This cannot be solved analytically and when an analytical solution for maximizing the log-likelihood function (To find the maximum likelihood estimates or MLES) is not possible, numerical optimization algorithms are used. These iterative methods approximate the parameter values that maximize the function.

VOLATILITY MODELLING

The volatility models considered in the study are the GJR-GARCH and EGARCH.

Glosten Jaggannathan and Runkle Generalized Autoregressive Conditional Heteroscedasticity (GJR-GARCH) model.

$$\sigma_t^{2}=\phi+\sum_{i=1}^{p}\left(\alpha_i\varepsilon_{t-1}^2+\gamma_1 I_{t-i}\varepsilon_{t-1}^2\right)+\sum_{j=1}^{q}\beta_j\sigma_{t-i}^2.$$

$$\varepsilon_t=\sigma_t z_t \tag{24}$$

$$I_{t-i}=\begin{cases}1,\varepsilon_{t-i}<0\\ 0,\varepsilon_{t-i}\geq 0\end{cases}$$

ϕ is constant term (long-run volatility) α_i is ARCH term while β_j is the GARCH term, γ_1 is the leverage term, $\phi\geq 0$, α_i and $\beta_j\geq 0$ and σ_t is the volatility.

Exponential Generalized Autoregressive Conditional Heteroscedasticity (EGARCH) model

$$r_t=\mu+\varepsilon_t$$

$$\ln(\sigma_t^2)=\phi+\sum_{i=1}^{p}\alpha_i\left[\lambda\varepsilon_{t-i}+\gamma\left\{\left|\varepsilon_{t-i}\right|-\sqrt{\frac{2}{\pi}}\right\}\right]+\sum_{j}^{q}\beta_j\ln(\sigma_{t-j}^2) \tag{25}$$

where, ϕ is constant term, α_i is ARCH term while β_j is the GARCH term and, γ is the leverage term and σ_t is the volatility.

ARCH effect test

To test for ARCH effect, the Lagrange Multiplier test was employed. The Lagrange Multiplier test expressed the error terms as a function of the past error terms as expressed below:

$$\varepsilon_t^2=\omega_0+\omega_1\varepsilon_{t-1}^2+\ldots+\omega_p\varepsilon_{t-p}^2+a_t \tag{26}$$

The null and alternative hypotheses are:

$H_0:\omega_1=\ldots=\omega_p=0$ versus

$H_1:\omega_i\neq 0$ for some $i\in\{1,..,p\}$

Simulation Study

We generated 1,000 observations (T = 1,000) from EGARCH (1,1) and GJR-GARCH (1,1) models. The innovation series were drawn from several distributions, including Student-t, generalized error distribution (GED) and exponentiated Student-t (ESTD). The asymmetric GARCH models were parameterized with $\omega = -0.1$, $\alpha = 0.15$, $\gamma = 0.1$, and $\beta = 0.9$. For the Student-t innovations, the degrees of freedom were set to $\nu = 8$, while for the GED innovations, the shape parameter was set to $\nu = 1.5$. All simulations were conducted in Python using the arch package (Sheppard, 2016). To ensure reproducibility, the random number generator was initialized with a fixed seed (np.random.seed(42)), so that each run produces the same simulated series. Each simulation design was replicated 1,000 times to evaluate the finite-sample properties of the parameter estimators.

Model Selection

The fitness of the proposed exponentiated distribution within the asymmetric GARCH models considered was assessed using the log- likelihood function, Akaike Information Criteria (AIC) and Bayesian Information Criteria (BIC). AIC is defined by:

$$AIC = -\frac{2\log(LL) + 2\beta}{n} \quad (27)$$

where, LL is the likelihood of the model and β is the number of parameters in the model and n is the number of observations.

BIC = - 2 ln (L) + K + In(n) (28)
Where L is the maximum value of the likelihood function for the model, K is the number of parameters in the model and n is the number of observations.

Forecasting Evaluation
the forecasting evaluation was done using the performance metrics such as the Means Square Error (MSE), Root Mean Square Error (RMSE) and Mean Absolute Error (MAE) as recommended by Frances [15] and Khair et al [16].

$$MSE = \frac{1}{n-1}\sum_{t=2}^{n}\left(\sigma_t^2 - \hat{\sigma}_t^2\right)^2 \quad (29)$$

$$RMSE = \sqrt{\frac{1}{n-1}\sum_{t=2}^{n}\left(\sigma_t^2 - \hat{\sigma}_t^2\right)^2} \quad (30)$$

$$MAE = \frac{1}{n-1}\sum_{t=2}^{n}\left|\sigma_t^2 - \hat{\sigma}_t^2\right| \quad (31)$$

APPLICATION TO DATA

This section illustrates the flexibility of the new exponentiated student -t distribution when compared to some existing distribution in volatility modeling of financial data. The daily log returns of the United States of America stock - S&P 500 stock index was used to determine the flexibility of this error distribution. Daily data of S&P 500 stock index from 1st January,2005 to 1st August, 2025 from https://finance.yahoo.com. This data set was converted to daily log returns using the formular:

$$r_t = \ln\left(\frac{P_t}{P_{t-1}}\right) X100 \qquad t = 2,3, \ldots, n \quad (32)$$

where, r_t is the daily returns, P_t is the daily closing price at the present day while P_{t-1} is the daily closing price at the previous day.

RESULTS AND DISCUSSION

The result of the descriptive statistics as shown in Table 1 showed that the mean return series is positive, the skewness is negative indicating that the returns fall more than they rise and high kurtosis indicating consistent fluctuation away from the average return of the series.

Table 1: Descriptive Statistics of Log Return of S&P 500 Stock Index
Descriptive Statistics

Mean	0.000321
Standard Deviation	0.012765
Skewness	-0.4735
Kurtosis	16.1045
Jarque-Bera	37229.4314
P – Values	0.0000
Observation	5176

Stationarity Test
The statistic of the ADF in Table 2 shows that the series was stationary since the ADF statistic is greater than 1% critical level. Therefore, there was no need for transformation.

Table 2: Augmented Dickey-Fuller Test of Stationarity (ADF) test of S&P 500 Stock Index

Series	ADF Test Statistic	P-value	Comment
S&P 500 Stock Index	- 16.1472	0.0000	Stationary

ARCH Effect Test
The ARCH effect was found to be significantly present in the return series since ($p < 0.05$) as shown in Table 3

Table 3: ARCH Effect Test

Series	ARCH Test (F statistics)	P -Value	Comment
S&P 500 Index	1483.1496	0.0000	Evidence of ARCH Effect

Parameter Estimates of the Volatility Models based on simulated Result

Table 4: Parameter estimates Asymmetric GARCH models based on the existing and the proposed distribution with simulated data.

Models	Innovation distributions	ϕ (p-values)	α (p-value)	β (p-value)	γ (p-value)	Shape (c) (p-value)	Volatility persistence
EGARCH (1,1)	Student-t	0.6827 (0.0035)	0.1591 (0.1396)	0.9171 (0.0001)	0.1011 (0.0033)	-	0.9171
	GED	0.3376 (0.0240)	0.1009 (0.0108)	0.9586 (0.0009)	0.0463 (0.0419)	-	0.9586
	ESTD	0.5151 (0.0412)	0.0831 (0.0416)	0.9349 (0.0000)	0.0633 (0.0653)	1.5961 (0.0000)	0.9349
GJR GARCH (1,1)	Student-t	1316.04 (0.0127)	0.1050 (0.0008)	0.7954 (0.0001)	0.1057 (0.0659)	-	0.9533
	GED	768.83 (0.0010)	0.1415 (0.0009)	0.7661 (0.0001)	0.1848 (0.0041)	-	1.0000
	ESTD	404.67 (0.0057)	0.1296 (0.0020)	0.8577 (0.0000)	0.1078 (0.0147)	(1.0028) 0.0000	0.9334

Simulations with **Exponentiated Student-t (ESTD) innovations** for both EGARCH (1,1) and GJR-GARCH (1,1) models produced parameter estimates consistent with **asymmetric volatility dynamics**. In EGARCH (1,1), the short-term volatility parameter α was 0.0831, the leverage effect γ was positive at 0.0633, and long-run persistence (α + β) was 0.9349. In GJR-GARCH (1,1), α was 0.1296, γ was positive at 0.1078, and volatility persistence reached 0.9334. The shape parameter for ESTD was 1.5961 (EGARCH) and 1.0028 (GJR-GARCH), confirming the heavy-tailed nature of the innovations. **Statistically,** positive γ indicates that **negative shocks increase conditional volatility more than positive shocks of the same magnitude**, while high β ensures that these effects persist over time. **Economically,** this reflects the classic leverage effect in financial markets: large losses trigger stronger volatility responses than gains, capturing the heightened risk following negative returns. ESTD's heavy tails increase the impact of extreme negative events, while its skewness allows the model to differentiate between extreme positive and negative shocks. This behavior is particularly relevant for **portfolio risk management, Value-at-Risk estimation, derivative pricing, and stress-testing,** as it provides a realistic representation of downside risk and prolonged volatility clustering. Overall, ESTD with positive γ demonstrates both **statistical robustness and economic realism**, making it a compelling choice for modeling financial returns with asymmetric and extreme shocks.

Parameter Estimates of the Volatility Models based on S &P 500 Stock Index

Table 5: Parameter estimates of asymmetric GARCH models based on the existing and proposed distribution with S& P 500 data.

Models	Innovation distributions	ϕ (p-values)	α (p-value)	β (p-values)	γ (p-value)	Shape (c) (p-value)	Volatility Persistence
EGARCH (1,1)	Student-t	0.0034 (0.0100)	0.1672 (0.0001)	0.9775 (0.000)	-0.1791 (0.0007)	-	0.9775
	GED	0.004 (0.0030)	0.1812 (0.0008)	0.9742 (0.000)	-0.1632 (0.0002)	-	0.9742
	ESTD	0.0114 (0.0013)	0.1443 (0.0000)	0.9705 (0.0000)	0.1526 (0.0000)	1.0397 (0.0000)	0.9705
GJR-GARCH (1,1)	Student-t	0.0214 (0.0004)	0.0000 (0.0199)	0.8590 (0.0000)	0.2460 (0.0006)	-	0.9820
	GED	0.0214 (0.0000)	0.00001 (0.0099)	0.8590 (0.000)	0.2460 (0.0000)	-	0.9821

	ESTD	0.0166 (0.0004)	0.0007 (0.0312)	0.8577 (0.0000)	0.1770 (0.0000)	1.0419 (0.0000)	0.9469

Table 5 presents the parameter estimates of asymmetric GARCH models based on the existing and proposed exponentiated student -t distribution (ESTD). The parameter estimates of the volatility models were estimated using three innovation distributions namely student -t distribution, generalized error distribution (GED) and Exponentiated student -t distribution (ESTD). The result shows that the parameter estimates are statistically significant at the 5% level, the constant term (ϕ) is positive and statistically significant (p -value < 0.05) in both models EGARCH (1,1) and GJR- GARCH (1,1). This indicates that there is a persistent, non-zero level underlying variance in the absence of recent specific market shocks. This corresponds with the work of [17] who used GARCH- type models to examine the volatility in commodity of stock returns and reported persistent volatility in financial series. There is presence of ARCH (α) effect in both models since the p -value < 0.05. The positive and significant α coefficient confirms the presence of volatility clustering in the S&P 500 returns. This means that large shocks (both positive and negative, to varying degrees) tend to be followed by more large shocks, and small shocks tend to be followed by more small shocks, meaning that recent past shocks are primary driver of current market volatility. A positive and significant GARCH term (β) coefficient confirms that past periods of high or low volatility have a persistent effect on future volatility. The significance indicates that volatility is persistent and does not revert to its long-term average immediately after a shock. This has major implications for risk management and asset pricing of S&P 500. There is evidence of leverage effect (γ) in both models which indicates that positive shocks (good news) increase market volatility more than negative shocks (bad news) of the same magnitude**.** The additional shape parameter (a) is significant in both EGARCH (1,1) and GJR- GARCH (1,1) models. This show that the additional shape parameter in the proposed error distribution provides a more realistic and robust representation of the S&P 500's risk profile, especially regarding extreme events and asymmetry, which is vital for accurate risk management and asset pricing.

The persistence measures under the ESTD specification remain below unity and are comparable to those obtained under GED and Student-t assumptions. This indicates that the improved likelihood performance of the ESTD model is not driven by artificial inflation of volatility persistence but rather by enhanced flexibility in modeling return innovations.

Table 6: Fitness performance evaluation of the different Asymmetric GARCH models on S & P 500

Models	Innovation distribution	LL	AIC	BIC
EGARCH (1,1)	Student-t	5246.65	-10505.3	-10543.3
	GED	5256.04	-10524.1	-10562.1
	ESTD	5261.23	-10534.46	-10572.43
GJR-GARCH (1,1)	Student-t	5252.78	-10517.6	-10555.5
	GED	5276.64	-10563.29	-10584.93
	ESTD	5281.23	-10574.46	-10592.43

Table 6 presents the fitness performance evaluation of the different asymmetric GARCH models. The result of the fitness is based on log- likelihood (LL), Akaike Information criteria (AIC) and Bayesian information criteria (BIC) of EGARCH (1,1) and GJR- GARCH models with error distributions, student -t, GED and ESTD. The result indicates that ESTD fitted best the volatility of S& P 500 as it produces the maximum log-likelihood and least values of AIC and BIC for EGARCH (1,1) and GJR-GARCH (1,1). This finding is in agreement with the work of [18] that EGARCH model is the preferred one of intraday volatility estimation in S&P500 stock index future product. Comparatively, GJR-GARCH (1,1) exhibits a slightly better fit than EGARCH (1,1), reflecting its ability to capture asymmetric volatility responses more effectively. Economically, the ESTD enhances the models' capacity to account for extreme market events and fat-tailed behavior, providing more accurate volatility estimates. This has significant implications for financial risk management and derivative pricing, as the model is better suited to respond to negative shocks and tail risks. Overall, the combination of GJR-GARCH (1,1) with ESTD represents the most reliable specification for modeling financial time series volatility, highlighting the practical advantages of the proposed innovation over conventional distributions.

Table 7: Out of sample forecasting performance evaluation of the different Asymmetric GARCH model based on S & P 500

Models	Innovation distributions	MSE	RMSE	MAE
EGARCH (1,1)	Student-t	0.000000112947	0.0003360758	0.0001299966
	GED	0.000000113041	0.0003362170	0.0001283508
	ESTD	0.000000112681	0.0003356799	0.0001143894
GJR-GARCH(1,1)	Student-t	0.000000112706	0.0003357184	0.0001323757
	GED	0.000000112353	0.0003351910	0.0001296351
	ESTD	0.000000112349	0.0003351850	0.0001159408

Table 7 presents the Out of sample forecasting performance evaluation of the different asymmetric GARCH model on S & P 500. The result reveal that the error distribution EGARCH (1,1) and GJR-GARCH with ESTD has the least RMSE and MAE. These findings corroborate with the findings of [19] whose research showed that the GJR-GARCH model often produces more accurate forecasts for the S&P 500 index, while the EGARCH was the second-best model. This also agreed with the work of [20] on modeling volatility of S&P 500 index daily returns that GJR-GARCH out performs best in terms of fitting and forecasting performance. This finding did not agree with the findings of Agboola et al 2018, their result showed that TGARCH and APARCH model outperformed in terms of best fitness and forecasting performance under the proposed error innovation distribution, this differences could be as a result of the time period analyzed (e.g., during a crisis vs. calm periods), and whether the focus is on in-sample fit or out-of-sample forecasting ability. From an economic perspective, these findings suggest that the ESTD innovation provides greater flexibility in capturing the heavy-tailed behavior and extreme fluctuations commonly observed in S&P 500 return series. This improved representation of market dynamics enhances the accuracy of volatility forecasts, which is essential for financial risk management, derivative pricing, and portfolio allocation. Overall, the empirical evidence demonstrates that incorporating the proposed ESTD innovation into asymmetric GARCH models significantly improves both model's fit and forecasting performance, making it a more reliable framework for analyzing and predicting volatility in S&P 500 financial market.

CONCLUSION

This study proposed a new error innovation distribution, the Exponentiated Student-t Distribution (ESTD)**,** for improving volatility modeling within asymmetric GARCH frameworks. The performance of the proposed innovation was evaluated against conventional distributions, namely the student-t and the Generalized Error Distribution (GED), using the EGARCH (1,1) and GJR-GARCH (1,1) models. The models were assessed using both simulated data and empirical financial data based on the returns of the S&P 500 index. The empirical findings consistently demonstrate that the ESTD innovation outperforms the benchmark distributions in both in-sample model estimation and out-of-sample forecasting performance. Specifically, the ESTD-based models produced higher log-likelihood values and lower information criteria (AIC and BIC), indicating superior goodness-of-fit. Furthermore, the out-of-sample forecasting results based on Mean Squared Error (MSE), Root Mean Squared Error (RMSE), and Mean Absolute Error (MAE) show that models incorporating the ESTD innovation generate the most accurate volatility forecasts. These results suggest that the additional flexibility introduced by the exponentiation mechanism enables the ESTD to better capture the heavy-tailed and extreme fluctuations commonly observed in financial return series. Consequently, the proposed distribution provides a more reliable framework for modeling financial market volatility

The main contribution of this study lies in the development and empirical validation of a new innovation distribution that enhances the performance of asymmetric GARCH-type volatility models. By demonstrating the superior statistical fit and forecasting accuracy of the ESTD innovation across both simulated and real financial data, the study provides evidence that incorporating more flexible innovation distributions can significantly improve volatility modeling and financial risk assessment.

Despite these promising results, the study has certain limitations. First, the empirical analysis was restricted to a single financial market index, which may limit the generalizability of the findings across different financial markets or asset classes. Second, the study focused only on two asymmetric GARCH specifications, namely EGARCH and GJR-GARCH, while other volatility models may exhibit different interactions with the proposed distribution. Third, the analysis was conducted within a univariate framework, which does not capture potential cross-market volatility spillovers.

Future research may extend this work in several directions. Further studies could apply the ESTD innovation to other financial markets, including emerging markets, foreign exchange markets, and cryptocurrency returns, to evaluate its robustness across different financial environments. Additionally, the proposed distribution could be incorporated into alternative volatility models such as APARCH, stochastic volatility models, or multivariate GARCH frameworks to investigate its performance in more complex settings. Finally, future work may explore the application of the ESTD innovation in financial risk management tools such as Value-at-Risk and expected shortfall estimation.

Overall, the results of this study demonstrate that the proposed ESTD innovation provides a flexible and effective alternative for modeling financial market volatility and offers promising potential for improving volatility forecasting and risk management practices.

REFERENCE

Su, C. (2010). Application of EGARCH model to estimate returns, the empirical case of China, M.Sc. Thesis, University of Gothenburg, Sweden.

Raheem, M. A., & Timothy, S. K. (2020). Volatility modeling of global financial crises effects on Nigerian banks. *Nigerian Bank Journal*, **10**(2), 303–324.

Timothy, K. S., Christian, E. O., & Adedoyin, I. L. (2023). An examination of cryptocurrency volatility: Insights from skewed error innovation distributions within GARCH model frameworks. *Mathematical Modeling of Engineering Problems*, **10**(1), 1297–1306.

Engle, R. F. (1982). Autoregressive conditional heteroskedasticity with estimates of the variance of United Kingdom inflation. *Econometrica*, **50**(4), 987–1007.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics*, **31**(3), 307–327.

Nelson, D.B. (1991). Conditional heteroscedasticity in asset returns: A New Approach. Econometrica,59 (2),347-370.

Yaya, O.S, (2013). Nigerian stock index: A search for optimal GARCH model using high frequency data. *Central Bank of Nigeria Journal of Applied Statistics*, 4(2), 69-85.

Onwukwe, C.E, Samson, T.K & Lipcsey, Z. (2014). Modeling and forecasting daily return volatility of Nigerian banks stocks. *European Scientific Journal, 10 (15): 449-467.*

Samson, T.K., Onwukwe, C.E. & Enang, E.I. (2020). Modeling volatility in Nigerian stock market: Evidence from skewed error distributions. *International Journal of modern Mathematical Science*, 18 (1), 42-57

Agboola, S., Dikko, H. & Asiribo, O. E (2018). On exponentiated skewed student -t error distribution on some heteroscedastic models: Evidence of Nigerian stock exchange, Journal of statistics application and probability 7, 233-244

Agboola, S., Dikko, H. & Asiribo, O. E (2018). On exponentiated skewed student -t error distribution on some heteroscedastic models: Evidence of Nigerian stock exchange, *Journal* *of statistics application and probability* 7; 233-244.

Gupta, R.C, Gupta, R.D & Gupta, P.L (1998). Modeling failure time data by Lehman alternatives communications in statistics - theory and methods, 27(4), 887-904.

Eghwerido, J. T., Ikwuoche, J. D., & Adubisi, O. D. (2020). Inverse odd Weibull generated family of distribution. *Pakistan Journal of Statistics and Operations Research*, **16**(4), 617–633.

Adubisi, O. D., Abdulkadir, A., Farouk, U. A., & Chiroma, H. (2022). The exponentiated half logistic skew-t distribution with GARCH-type volatility models. *Scientific African*, **16**, e01253. https://doi.org/10.1016/j.sciaf.2022.e01253

Franses, P. (2016). A note on the Mean Absolute Scaled Error. International Journal of Forecasting, 32 (1), 20-22.

Kahair, U., Fahmi, H., Hakim, S, & Rahim, R. (2017). Forecasting error calculation with mean absolute deviation, mean absolute percentage error. *Journal of Physics: conference series*,930, 1-6.

Zubair, M. A., & Boyi, B. A. (2025). *Modelling the volatility of cotton and rubber stock returns in Nigeria using GARCH type models.* ***FUDMA Journal of Sciences, 9****(5), 67–79. https://doi.org/10.33003/fjs-2025-0905-3650*

Xie, H., & Li, J. (2010). Intraday volatility analysis on S&P 500 stock index futures. *International Journal of Economics and Finance*, **2**(2), 45–53.

Liu, H.C & Hung J.C (2020). Forecasting S& P 100 stock index volatility: The role of Volatility of asymmetry and distributional assumption in GARCH models.,36 (4), 60-70

Huang, A. Y. (2011). Volatility modeling by asymmetrical quadratic effect with diminishing marginal impact. *Computational Economics*, **37**(3), 301–330.