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Elzaki Based Hybrid Method for The Solutions of Nonlinear Integer Order Differential Equations

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ABSTRACT
This paper is concerned with the solutions of nonlinear differential equations by a proposed hybrid method that is developed through a well-considered integration of a Laplace-type integral transform; Elzaki transform into a semi-analytical method called Adomian decomposition method. This hybrid approach is tagged Elzaki Based Hybrid Method (EBHM). To apply the new method, Elzaki transform of the terms in the given nonlinear differential equations is first taken, while Adomian polynomial is used to decompose the nonlinear term that may be encountered. To get the desired result, inverse Elzaki transform is made use of at the final stage of the computation. To validate the proposed method, it is used to solve selected problems in the existing literature. The method gives exact solutions for all the problems considered with reduced computational volume. All the coding and computations are carried out with Mathematica 13.3.

Keywords:
Integral transform,
Decomposition Method,
Adomian Polynomials,
Hybrid Method

INTRODUCTION

The importance of differential equations (DEs) cannot be overemphasized. DEs can be classified as either ordinary differential equations (ODEs) or partial differential equations (PDEs) based on the independent variables (Tadmor, 2012). Most real-world applications to equations are not the regular linear and constant coefficient equations that can be solved by traditional analytical methods (Ganie, 2024), (ELZAKI, 2025), (Idrees, 2018), (Sanda, 2026), and (Delic, 2023). Numerical approaches are sometimes used but gives approximate solutions, an example of which is Runge-Kutta method of order four (Lawal, Baoku, & Yusuf, 2026). Integral transform methods are used in solving initial value problems (IVPs) related to integral equations, integro-differential equations, and ODEs (Al-Bugami, 2025) and (Jafari, 2021). The Laplace transform, introduced by Pierre-Simon (Schiff, 1999), is effective for linear and constant coefficient equations but it breaks down whenever the problem has variable coefficients. The Sumudu transform, proposed by (Watugala, 1993), also suffers the same limitation as the Laplace transform (Elzaki, 2012). Other examples are the Mohand, Fourier, Mahgoub, and Aboodh transforms, amongst others.

The Elzaki transform was proposed and reported in (Elzaki, 2011b). This transform builds on the Laplace and Sumudu transforms, uses a more flexible integral kernel, and is useful for higher-order and variable-coefficient differential equations (Elzaki, 2011b).

The Elzaki transform works well for solving ordinary, partial, and integro-differential equations Semi-analytical approaches balance between analytical and numerical methods by providing approximate analytical solutions.

An example is the Adomian Decomposition Method (ADM) proposed by George Adomian in 1988, which can directly solve nonlinear differential equations (Adomian, 1994). This also has its own limitations as it converges slowly when applied to equations with complex or strong nonlinear differential equations (Kumar, 2022) (Agbata, 2022).

In a bid to overcome the limitations of integral transforms and semi-analytical methods, hybrid analytical methods have been developed. These methods integrate integral transforms into semi-analytical methods in a bid to solve complex nonlinear differential equations. An example of this is the Elzaki Based Hybrid Method (EBHM) that is proposed in the present study. This method has proven to be reliable and efficient for solving nonlinear, variable coefficient, and fractional order differential equations. This analytical method offers better convergence, less computational effort, and greater adaptability than the Adomian decomposition method, being hybridized with the earlier integral transforms.

Preliminaries

The Elzaki Transform

Definition (Elzaki, 2011b)

The Elzaki transform of the function $f(\sigma)$ is defined as

$$E\{f(\sigma)\} = F(\tau) = \tau \int_0^\infty f(\sigma)\, e^{\left\{-\frac{\sigma}{\tau}\right\}}\, d\sigma.$$

The inverse Elzaki transform, denoted by $E^{-1}[.]$ is given by

$$E^{-1}[F(\tau)] = f(\sigma).$$

Elzaki Transform of Derivatives

Let $F(\tau)$ be the Elzaki transform of $[E(f(\sigma)) = F(\tau)]$, then:

i. $E\{f'(\sigma)\} = \frac{F(\tau)}{\tau} - \tau f(0)$

ii. $E\{f''(\sigma)\} = \frac{F(\tau)}{\tau^2} - f(0) - \tau f'(0)$

iii. $E\{f'''(\sigma)\} = \frac{F(\tau)}{\tau^3} - \frac{f(0)}{\tau} - f'(0) - \tau f''(0)$

The general formula for the Elzaki transform of the nth derivative is defined by

$$E\{f^{(n)}(\sigma)\} = \frac{F(\tau)}{\tau^n} - \sum_{i=0}^{n-1} \tau^{2-n+i} \sigma^{(i)}(0)$$

Elzaki Transform of Some Mathematical Expressions

In this section, the application of Elzaki integral transform to a few mathematical expressions is presented.

Example 1

Find the Elzaki transform of $f(\sigma) = 1$.

Solution

When $f(\sigma) = 1$, we have

$$E\{1\} = \tau \int_0^\infty e^{\frac{\sigma}{\tau}}\, d\sigma$$

$$E\{1\} = \tau \left[-\tau \cdot \left(\frac{1}{e^{\frac{\sigma}{\tau}}}\right)\right]_0^\infty$$

$$E\{1\} = \tau^2 \left[\frac{1}{e^{\frac{\sigma}{\tau}}}\right]_0^\infty$$

$$E\{1\} = \tau^2$$

Example 2

Find the Elzaki transform of $f(\delta) = a$, where a is a constant.

Solution

When $f(\delta) = a$, we have

$$E\{a\} = \tau \int_0^\infty a\, e^{\left\{-\frac{\sigma}{\tau}\right\}}\, d\sigma$$

$$E\{a\} = a\tau \left[-\tau \cdot \left(\frac{1}{e^{\frac{\sigma}{\tau}}}\right)\right]_0^\infty$$

$$E\{a\} = a\tau^2 \left[\frac{1}{e^{\frac{\sigma}{\tau}}}\right]_0^\infty$$

$$E\{a\} = a\tau^2$$

Example 3

Find the Elzaki transform of $f(\sigma) = e^{at}$

Solution

When $f(\sigma) = e^{at}$, we have

$$E\{e^{at}\} = \tau \int_0^\infty e^{at}.\, e^{\left\{-\frac{\delta}{\tau}\right\}}\, d\sigma$$

$$E\{e^{at}\} = \tau \left[\frac{1}{1/\tau - a} e^{-\tau(\frac{1}{\sigma} - a)}\right]_0^\infty$$

$$E\{e^{at}\} = \tau \left(\frac{\tau}{1 - a\tau}\right)$$

$$E\{e^{at}\} = \frac{\tau^2}{1 - a\tau}.$$

	Function $f(\sigma)$	Elzaki Transform $E(f(\sigma)) = F(\tau)$
1.	1	τ^2
2.	t	τ^3
3.	t^n	$n!\,\tau^{n+2}$
4.	e^{at}	$\frac{\tau^2}{1 - a\tau}.$
5.	$\sin(at)$	$\frac{a\tau^3}{1 + a^2\tau^2}$
6.	$\cos(at)$	$\frac{\tau^2}{1 + a^2\tau^2}$
7.	$\sinh(at)$	$\frac{a\tau^3}{1 - a^2\tau^2}$
8.	$\cosh(at)$	$\frac{\tau^2}{1 - a^2\tau^2}$

Elzaki Transform of Some Variable Functions

1. $E\{\sigma f'(\sigma)\} = \tau \int_0^\infty \sigma f'(\sigma) e^{-\frac{\sigma}{\tau}} d\sigma$

$$= -\tau \int_0^\infty f'(\sigma) e^{-\frac{\sigma}{\tau}} d\sigma + \int_0^\infty \sigma f'(\sigma) e^{-\frac{\sigma}{\tau}} d\sigma$$

But $F(\tau) = \tau \int_0^\infty f(\sigma)\, e^{\left\{-\frac{\sigma}{\tau}\right\}}\, d\delta$ and $E\{\sigma f(\sigma)\} = \tau \int_0^\infty \sigma f(\sigma) e^{-\frac{\sigma}{\delta}} d\sigma$.

This implies that

$$E\{\sigma f'(\sigma)\} = -F(\tau) + \frac{E\{\sigma f'(\sigma)\}}{\tau}$$

$$E\{\delta\sigma(\sigma)\} = \tau F'(\tau) - 2F(\tau).$$

2. $E\{\sigma f'(\sigma)\} = F'(\tau) - \frac{3F(\tau)}{\tau} - \tau f(0).$

3. $E\{\sigma^2 f''(\sigma)\} = \tau^4 \frac{d^2}{d\tau^2}\left[\frac{F(\tau)}{\tau^2} - f(0) - \tau f(0)\right]$

A Brief Review of Adomian Decomposition Method

This section reviews the Adomian Decomposition Method as it is the semi-analytic method combined with the Elzaki Integral Transform.

Definition

The Adomian Decomposition Method (ADM) is a semi-analytical technique for solving ordinary and partial differential equations, either linear or non-linear. It decomposes the unknown function $y(x,t)$ into a sum of an infinite series:

$$y(x,t) = \sum_{n=0}^{\infty} y_n(x,t)$$

The non-linear terms in the equation are represented by a specific set of polynomials known as Adomian Polynomials. The standard formula for generating Adomian polynomials for a nonlinear function $y(x)$ is:

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^{n} \lambda^i y_i\right)\right]_0^{\infty},$$

where λ is an auxiliary grouping parameter used to sort the terms by their order of magnitude (Wazwaz, 2000).

Consider a general non-linear differential equation in the operator form:

$$Ly(x) + Ry(x) + N\big(y(x)\big) = g(x), \qquad (1a)$$

with the initial conditions,

$$y(0) = \beta_1, \quad y'(0) = \beta_2, \cdots \ , y^{(n-1)}(0) = \beta_n \quad (1b)$$

where:

L is the highest-order differential operator,

R is the remaining linear term,

N is the non-linear operator, and,

$g(x)$ is the source term.

$$Ly(x) \ = \ g(x) \ - Ry(x) - \ N\big(y(x)\big)$$

Applying the inverse operator, L^{-1} to the above equation, we have

$$y(x) \ = \ L^{-1}[g(x)] - L^{-1}[Ry(x)] - L^{-1}[N\big(y(x)\big)]$$

Assume that the solution $y(x)$ can be represented as an infinite series of components,

$$y(x) = \sum_{n=0}^{\infty} y_n(x)$$

Then,

$$\sum_{n=}^{\infty} y_n(x) = \ L^{-1}[g(x)] - L^{-1}[Ry(x)] - L^{-1}[N\big(y(x)\big)]$$

$$y_0(x) = \psi_0(x) + \propto_0 (x),$$

where,

$$\psi_0(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \cdots$$

and

$$\propto_0 (x) = L^{-1}[g(x)].$$

Then,

$$y_{n+1}(x) = -L^{-1}[Ry(x)] - L^{-1}[A_n(x)] \qquad (2)$$

where $A_n(x)$ are the Adomian polynomials derived for the nonlinear terms in the IVP. The Adomian Polynomials are obtained from.

Statement of the Problem

Consider the nth order inhomogeneous nonlinear differential equation

$$Ly(x) + N[y(x)] = g(x), \qquad (3a)$$

with the associated initial conditions,

$$y(0) = \beta_1,$$
$$y'(0) = \beta_2, \ \cdots \ , y^{(n-1)}(0) = \beta_n, \qquad (3b)$$

where

L is a nth order linear differential operator,

N is the nonlinear operator, and

$f(x)$ is a smooth function called inhomogeneous source term

MATERIALS AND METHODS

Application of Elzaki Transform to Ordinary and Partial Differential Eqauations

Consider the nth order inhomogeneous linear differential equation

$$Ly(x) = f(x), \qquad (4a)$$
$$y(0) = \propto_1, \quad y'(0) = \propto_2, \cdots \ , y^{(n-1)}(0) = \propto_n \qquad (4b)$$

where

L is a nth order linear differential operator,

$f(x)$ is a known function otherwise referred to as inhomogeneous source term, and

$\propto_1, \propto_2, \cdots, \propto_n$ are the initial conditions given.

From (4a),

$$L = \frac{d^n}{dx^n} + \frac{d^{n-1}}{dx^{n-1}} + \ \cdots + \ \frac{d}{dx} + 1$$

So, applying Elzaki to both sides of (4a), we have

$$E\{Ly(x)\} = \ E\{fy(x)\}$$
$$E\left\{\frac{d^n}{dx^n} + \frac{d^{n-1}}{dx^{n-1}} + \ \cdots + \ \frac{d}{dx} + 1\right\} = E\{f(x)\}$$
$$\frac{F(\tau)}{\tau^n} - \sum_{i=0}^{n-1} \tau^{2-n+i} \sigma^{(i)}(0) = G(k)$$
$$F(\tau) = \tau^n \sum_{i=0}^{n-1} \tau^{2-n+i} \sigma^{(i)}(0) + \tau^n G(k) \qquad (5)$$

We then take inverse Elzaki of all terms on both sides of (5)

$$E^{-1}\{F(\tau)\} = E^{-1}\{\tau^n \sum_{i=0}^{n-1} \tau^{2-n+i} \sigma^{(i)}(0)\} + E^{-1}\{\tau^n G(k)\}, \qquad (6)$$

and (6) gives the desired result.

Method of Solution for the Class of Problem in (3)

To solve (3), we shall take the Elzaki transform of all terms in the given nonlinear differential equation:

$$E\{Ly(x)\} + \ E\{\ N[y(x)]\} = E\{g(x)\} \qquad (7)$$

But,

$$E\{Ly(x)\} = E\left\{\frac{d^n y}{dx^n} + \frac{d^{n-1}y}{dx^{n-1}} + \cdots + \frac{dy}{dx} + y\right\} \qquad (8)$$

$$E\{Ly(x)\} = \frac{F(\tau)}{\tau^n} - \sum_{i=0}^{n-1} \tau^{2-n+i} \sigma^{(i)}(0) \qquad (9)$$

$$E\{Ly(x)\} = \frac{F(\tau)}{\tau^n} - \frac{y(0)}{\tau^{n-2}} - \frac{y'(0)}{\tau^{n-1}} - \cdots - \tau y^{n-1}(0) \quad (10)$$

The equation (7) then becomes

$$\frac{F(\tau)}{\tau^n} - \frac{y(0)}{\tau^{n-2}} - \frac{y'^{(0)}}{\tau^{n-1}} - \cdots - \tau y^{n-1}(0) + E\{N[y(x)]\} = E\{g(x)\} \quad (11)$$

Using the initial points conditions,

$$F(\tau) = \tau^n G(\tau) + \tau^2\beta_1 + \tau\beta_2 + \cdots + \tau^{n+1}\beta_n - \tau^n E\{N[y(x)]\} \quad (12)$$

The nonlinear term $N[y(x)]$ shall be decomposed into a set of polynomials using Adomian decomposition, after which the inverse Elzaki of both sides will be taken to get the desired solution.

RESULTS AND DISCUSSION

Application of Elzaki Transform to IVPs in Ordinary Differential Equations

Problem 1

Solve the constant coefficient first order IVP

$$y' + 2y = x, \qquad y(0) = 1$$

Solution

Taking the Elzaki transform of each term, we have

$$E\{y'\} + 2E\{y\} = E\{x\}$$

$$\frac{F(\tau)}{\tau} - \tau + 2F(\tau) = \tau^3$$

$$F(\tau) = \frac{\tau^4 + \tau^2}{2\tau + 1}$$

Splitting the RHS of $F(\tau)$into partial fractions

$$F(\tau) = \frac{1}{2}\tau^3 - \frac{1}{4}\tau^2 + \frac{5\tau^2}{4(1+2\tau)}$$

Applying the inverse Elzaki transform gives

$$y(x) = \frac{1}{2}x - \frac{1}{4} + \frac{5}{4}e^{-2x}.$$

Problem 2

Solve the variable coefficient second order IVP

$$ty'' + 2y' = -\cos t, \qquad y(0) = 0, \quad y'(0) = \frac{1}{2}$$

Solution

Taking the Elzaki transform of each term, we have

$$E\{ty''\} + 2E\{y'\} = -E\{\cos t\}$$

$$T'(k) - \frac{T(k)}{k} = -\frac{k^2}{1+k^2}$$

Applying,

$$\frac{d}{dk}\left(I.F \;\; T(k)\right) = I.F \;\; Q(k)$$

$$I.F = \frac{1}{k}$$

$$\frac{d}{dk}\left(\frac{1}{\tau}F(\tau)\right) = -\frac{\tau}{1+\tau^2}$$

Integrating both sides,

$$F(\tau) = -\tau \, ln\sqrt{1+\tau^2}$$

Taking the series of $F(\tau) = -\tau \, ln\sqrt{1+\tau^2}$, we have

$$F(\tau) = -\tau\left(\frac{\tau^2}{2} - \frac{\tau^4}{4} + \frac{\tau^6}{6} - \frac{\tau^8}{8} + \frac{\tau^{10}}{10} + \cdots\right)$$

$$F(\tau) = -\frac{\tau^3}{2} + \frac{\tau^5}{4} - \frac{\tau^7}{6} + \frac{\tau^9}{8} - \frac{\tau^{11}}{10} + \cdots$$

Taking the inverse Elzaki of every term gives

$$y(t) = \frac{\cos(t) - 1}{t}.$$

Application of Elzaki Transform to Partial Differential Equations

Problem 1

Solve the second order PDE

$$u_{tt} - 4u_{xx} = 0, \qquad u(x,0) = \sin\pi x, \qquad u_t(x,0) = 0.$$

Solution

Taking the Elzaki transform of each term, we have

$$E\{u_{tt}\} - 4E\{u_{xx}\} = E\{0\}$$

$$\frac{F(x,\tau)}{\tau^2} - \sin\pi x - 4F''(x,\tau) = 0$$

$$F''(x,\tau) - \frac{1}{4\tau^2}F(x,\tau) = \frac{-\sin\pi x}{4}$$

Solution of the homogeneous part,

$$F''(x,\tau) - \frac{1}{4\tau^2}F(x,\tau) = 0$$

Let $F(x,\tau) = e^{mx}$, thus the auxiliary equation becomes

$$m^2 + \frac{1}{\tau^2} = 0$$

$$m = \pm\frac{1}{2\tau}$$

Solution of the particular part,

$$F_p = A\, sin\pi x + B\cos\pi x$$

$$F'_p = -A\pi\cos\pi x + B\pi\sin\pi x$$

$$F''_p = -A\pi^2\sin\pi x - B\pi^2\cos\pi x$$

$$-\frac{1}{4}\, sin\pi x = F''_p(x,\tau) = \frac{1}{4\tau^2}F_p(x,\tau)$$

$$= -A\sin\pi x\left(\pi^2 + \frac{1}{4\tau^2}\right) - B\cos\pi x\left(\pi^2 + \frac{1}{4\tau^2}\right)$$

$$A = \frac{\tau^2}{4\pi^2\tau^2 + 1}, \qquad B = 0$$

$$F_p = \frac{\tau^2\sin\pi x}{1 + (2\pi)^2\tau^2}$$

$$F(x,\tau) = \frac{\tau^2\sin\pi x}{1 + (2\pi)^2\tau^2}$$

Taking the inverse Elzaki of all terms give

$$E^{-1}\{F(x,\tau)\} = \sin\pi x\, E^{-1}\left\{\frac{\tau^2}{1 + (2\pi)^2\tau^2}\right\}$$

$$u(x,t) = \sin\pi x\cos 2\pi t.$$

Application of the Developed Algorithm to Some Nonlinear Problems

Problem 1 (Hermann, 2016)

Solve the first order nonlinear homogeneous IVP

$$y'(x) + y(x)^2 = 0, \quad y(0)$$

Solution

Taking the Elzaki of every term in the given ODE

$E\{y'(x)\} + E\{y(x)^2\} = E\{0\}$

$$F(\tau) - \tau^2 + \tau E\{y(x)^2\} = 0$$

$$F(\tau) - \tau^2 + \tau E\{y(x)^2\} = 0$$

$$F(\tau) = \tau^2 - \tau E\{y(x)^2\}$$

$$F_0(\tau) = \tau^2$$

Taking the inverse Elzaki, we get

$$E^{-1}\{F_0(\tau)\} = E^{-1}\{\tau^2\}$$

$$y_0(x) = 1$$

Using the Adomian decomposition method, we have

$$\sum_{n=1}^{\infty} F_n(x) = -\tau E\left\{\sum_{n=1}^{\infty} A_{n-1}(x)\right\}$$

$$A_0 = y_0^2, \quad A_1 = 2y_0y_1, \quad A_2 = 2y_0y_2 + y_1^2$$

$$F_1(\tau) = -\tau\, E\{A_0(x)\}$$

$$F(\tau) = -\tau\, E\{y_0^2\}$$

$$F_1(\tau) = -\tau\, E\{1\}$$

$$F_1(\tau) = -\tau^3$$

$$E^{-1}\{F_1(\tau)\} = -E^{-1}\{\tau^3\}$$

$$y_1(x) = -x$$

$$F_2(\tau) = -\tau\, E\{A_1(x)\}$$

$$F_2(\tau) = -\tau\, E\{2y_0y_1\}$$

$$F_2(\tau) = -\tau\, E\{2(1)(-x)\}$$

$$F_2(\tau) = \tau E\{2x\}$$

$$F_2(\tau) = \tau^4$$

$$E^{-1}\{F_2(\tau)\} = E^{-1}\{\tau^4\}$$

$$y_2(x) = x^2$$

$$F_3(\tau) = -\tau\, E\{A_2(x)\}$$

$$F_3(\tau) = -\tau\, E\{2y_0y_2 + y_1^2\}$$

$$F_3(\tau) = -\tau\, E\{2(1)(x^2) + (-x)^2\}$$

$$F_3(\tau) = \tau E\{3x^2\}$$

$$F_3(\tau) = -6\tau^5$$

$$E^{-1}\{F_3(\tau)\} = E^{-1}\{6\tau^5\}$$

$$y_3(x) = -x^3$$

The solution is

$$y(x) = \sum_{n=0}^{\infty} y_n(x)$$

$$= 1 - x + x^2 - x^3 + \cdots$$

$$y(x) = \frac{1}{1+x}.$$

Problem 2 (Hermann, 2016)

Solve the second order nonlinear homogeneous IVP

$$y''(x) - y(x)^2 = 2 - x^4, \quad y(0) = y'(0) = 0$$

Solution

Taking the Elzaki transform of every term in the given ODE, we have

$$E^{-1}\{y''(x)\} - E^{-1}\{y(x)^2\} = E^{-1}\{2\} - E^{-1}\{x^4\}$$

$$F(\tau) - \tau^2 E^{-1}\{y(x)^2\} = 2\tau - 24\tau$$

$$F(\tau) = 2\tau - 24\tau^8 + \tau^2 E^{-1}\{y(x)^2\}$$

$$F_0(\tau) = 2\tau^4 - 24\tau^8$$

Taking the inverse Elzaki, we get

$$E^{-1}\{F_0(\tau)\} = E^{-1}\{2\tau^4\} - E^{-1}\{24\tau^8\}$$

$$y_0(x) = x^2 - \frac{x^6}{30}$$

Using the Adomian Decomposition Method, we have

$$F_n(\tau) = \tau^2 E\{A_{n-1}(x)^2\}, \qquad n = 0,1,2,\cdots,$$

where

$$A_{n-1}(x) = y_{n-1}(x)^2$$

$$N(y) = y^2$$

$$A_0 = y_0^2, \; A_1 = 2y_0y_1, \; A_2 = y_1^2 + 2y_0y_2, \; A_3 = 2y_0y_3 + 2y_2y_1, \quad etc$$

$$A_0 = \left(x^2 - \frac{x^6}{30}\right)^2$$

$$F_1(\tau) = F^2 E\{A_0\}$$

$$F_1(k) = \tau^2 E\left\{x^4 - \frac{x^8}{15} + \frac{x^{12}}{900}\right\}$$

$$F_1(k) = \tau^2\left(24\tau^6 - \frac{8!\,\tau^{10}}{15} + \frac{12!\,\tau^{14}}{900}\right)$$

$$F_1(k) = 24\tau^8 - \frac{8!\,\tau^{12}}{15} + \frac{12!\,\tau^{16}}{900}$$

Taking the inverse Elzaki transform of each term gives

$$E^{-1}\{F_1(\tau)\} = E^{-1}\{24\tau^8\} - E^{-1}\left\{\frac{8!\,\tau^{12}}{15}\right\} + E^{-1}\left\{\frac{12!\,\tau^{16}}{900}\right\}$$

$$y_1(x) = \frac{x^6}{30} - \frac{x^{10}}{1350} + \frac{x^{14}}{163900}$$

Since there exist noise terms in $y_0(x)$ and $y_1(x)$, the term left in $y_0(x)$ is the exact solution. That is,

$$y(x) = x^2.$$

Problem 3 (Hermann, 2016)

Solve the second order nonlinear inhomogeneous IVP

$$y''(x) - y'(x)^2 + y(x)^2 = 1, \quad y(0) = 1,\; y'(0) = 0$$

Solution

Taking the Elzaki transform of each term in the given ODE, we have

$$F(\tau) - \tau^2 - \tau^2 E\{\,y'(x)^2\} + \tau^2 E\{\,y(x)^2\} = \tau^4$$

$$F(\tau) = \tau^4 + \tau^2 + \tau^2 E\{\,y'(x)^2\} - \tau^2 E\{\,y(x)^2\}$$

$$F_0(\tau) = \tau^4 + \tau^2$$

Taking the inverse Elzaki, we have

$$y_0(x) = 1 + \frac{x^2}{2}$$

Using the Adomian decomposition method, we have

$$F_n(\tau) = \tau^2 E\{A_{n-1}(x)\} - \tau^2 E\{B_{n-1}(x)\}, \qquad n = 0,1,2,\cdots,$$

where $A_{n-1}(x) = y'_{n-1}(x)^2$ and $B_{n-1}(x) = y_{n-1}(x)^2$

$$A_1 = 2y_0'y_1', \quad A_2 = (y_1')^2 + 2y_2'y_0', \quad A_3 = 2y_3'y_0' + 2y_2'y_1'$$

Equivalently,

$$B_1 = 2y_0y_1, \quad B_2 = (y_1)^2 + 2y_2y_0, \quad B_3 = 2y_3y_0 + 2y_2y_1$$

Using the recurrence relation, we have

$y_1(x) = -\frac{x^2}{2} - \frac{x^6}{5!}$

$y_2(x) = -\frac{x^2}{8} - \frac{x^4}{120} - \frac{11x^8}{56(5!)} + \frac{168x^{10}}{10!}$

Due to the existence of noise terms in $y_0(x)$ and $y_1(x)$, the term left in $y_0(x)$ is the exact solution. That is,

$$y(x) = 1.$$

In the first part of the results, Elzaki transform was directly applied to solve initial value problems in ordinary differential equations, followed by the solutions to selected problems in partial differential equations in which cases we got the same result like those in (Elzaki, 2011a) and (Elzaki, 2011b) respectively. In the last examples on the nonlinear aspect, the newly developed method, EBHM is applied to selected nonlinear ordinary differential equations which are both homogeneous and inhomogeneous. The results obtained for the three problems considered are in perfect agreement with those in (Hermann, 2016) where ADM and VIM were used as methods of solution. A very interesting thing in the proposed method is the reduction in computation because of 'noise term' that exists in the $y_0(x)$ and $y_1(x)$ terms. In such situations, once the noise terms are removed from the $y_0(x)$, the remaining term(s) constitute the exact solution. 'Noise terms' are the same terms which differ only in sign. All the problems considered yielded exact solutions when Elzaki transform and Elzaki Based Hybrid Method were applied not linear and nonlinear IVPs respectively. ordinary and partial differential equations are considered; exact solutions are obtained in all cases.

CONCLUSION

Elzaki Based Hybrid Method (EBHM) has been applied to strongly nonlinear problems and exact solutions are obtained. The most attractive thing in the use of the method proposed in this, aside from the reduction in computation, is the fact that it generates exact solutions. This shows that the approach is applicable to most nonlinear problems in ordinary and partial differential equations.

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