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A Unifying Framework for Exponential–Gamma Related Lifetime Distributions with Special Cases and Extensions

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ABSTRACT
Lifetime and reliability modeling has motivated the development of flexible probability distributions capable of capturing diverse hazard rate behaviors. Two research streams have dominated the literature: parameter-driven generalizations of the Gamma distribution and mixture- or transformation-based Exponential–Gamma constructions. However, the lack of a unified framework limits systematic comparison and further extensions. This study presents a coherent framework that integrates these approaches, unifying classical and higher-order Exponential–Gamma distributions. The framework employs finite mixture constructions, product-type mechanisms, and transformation operators such as exponentiation and generalized exponentiation, yielding models including the Exponential, Gamma, Lindley, New Exponential–Gamma, Exponentiated New Exponential–Gamma, and Exponentiated Generalized New Exponential–Gamma distributions as special cases or extensions. Parameters were estimated using the Maximum Likelihood Method, and the models were applied to real-life lifetime data. Empirical evaluation based on −2 log-likelihood, AIC, and BIC demonstrates that higher-order Exponential–Gamma models outperform classical and related competing distributions. Key properties are derived in closed form, and graphical and numerical illustrations confirm the nesting structure and progressive increase in flexibility. These results highlight the practical usefulness of the framework and establish it as a coherent theoretical foundation for existing models, providing a flexible platform for developing new lifetime distributions.

Keywords:
Exponential–Gamma Distributions;
Lifetime modeling;
Mixture distributions;
Exponentiated Generalized Distributions;
Unifying framework

INTRODUCTION

The modelling of lifetime and reliability data continues to attract considerable attention because of its applications in engineering, medical sciences, actuarial studies, public health, insurance, and finance (Amiru et al., 2025). Traditional lifetime models such as the Exponential and Gamma distributions are widely applied due to their simplicity, yet they often fail to represent important data characteristics such as skewness, heavy tails, and non-monotonic hazard rate behaviour. These limitations become more pronounced when analyzing complex real-life failure mechanisms. The assumption of a constant hazard rate in the Exponential distribution limits its practical relevance, while the Gamma distribution, despite its flexibility, presents analytical challenges owing to the lack of closed-form expressions for some reliability measures (Lawless, 2011).

These limitations have driven the development of extended and generalized lifetime distributions capable of modelling a wide range of hazard rate shapes (Amiru et al., 2025; Gaddafi et al., 2025; Suleman et al., 2025; Umar & Yahya, 2021; Yahya & Umar, 2024; Zakariyau et al., 2025). One major research direction has focused on Gamma-based extensions, including the generalized Gamma distribution and its variants, which introduce additional shape parameters to enhance modelling flexibility (Stacy, 1962; Mead et al., 2017; Umar & Yahya, 2021). Modified forms of the generalized Gamma distribution have been shown to accommodate increasing, decreasing, bathtub-shaped, and unimodal hazard rates (Mead et al., 2017), although these models are primarily parameter-driven and do not explicitly incorporate Exponential–Gamma compounding mechanisms. Consequently, their structural connection to simpler lifetime models remains limited.

In parallel, a growing body of work has explored lifetime models constructed from combinations of the Exponential and Gamma distributions through mixture and compounding techniques (Lindley, 1958; Ogunwale et al., 2019; Umar & Yahya, 2021; Suleman et al., 2025; Zakariyau et al., 2025). The Exponential–Gamma distribution proposed by Ogunwale et al. (2019) demonstrated improved flexibility over the classical Exponential model while retaining analytical tractability (Umar et al., 2019a). Building on this foundation, Umar and Yahya (2021) introduced the New Exponential–Gamma distribution, which generalizes the Lindley distribution and exhibits superior goodness-of-fit properties across several real-life datasets. Further extensions through exponentiation led to the Exponentiated New Exponential–Gamma distribution, which provides enhanced control over tail behaviour and hazard rate shapes and has been shown to outperform several competing lifetime models using likelihood-based criteria (Zakariyau et al., 2025). These developments underscore the growing importance of Exponential–Gamma compounding as a viable modelling strategy.

Additional developments include more generalized and exponentiated forms of the New Exponential–Gamma distribution, incorporating multiple shape parameters and offering substantial flexibility in modelling lifetime data. These models have been evaluated using real datasets and compared with existing distributions based on measures such as the −2 log-likelihood, Akaike Information Criterion, and Bayesian Information Criterion, with results indicating improved performance (Suleman et al., 2025; Umar et al., 2019b; Yahya & Umar, 2024).). However, the proliferation of models has also increased conceptual fragmentation within the literature

Recent work by Aderoju et al. (2025) further extends this line of research. The authors introduced the New Extended Exponential–Gamma (NEEG) distribution, deriving its statistical properties and estimating parameters via the maximum likelihood method. The model was applied to COVID-19 datasets from Italy and Nigeria and benchmarked against competing lifetime distributions, including the Gamma, Exponential, UYEG, and variants of the Generalized Lindley distribution. Using information criteria such as AIC, AICc, BIC, and HQIC, alongside graphical density comparisons, the NEEG distribution consistently demonstrated superior goodness-of-fit and flexibility, particularly for skewed and heavy-tailed data. These findings reinforce the practical relevance of Exponential–Gamma related constructions in real-world lifetime modelling.

Despite the substantial progress in both Gamma-based generalizations and Exponential–Gamma constructions, the existing literature shows that these approaches have largely evolved independently. Limited attention has been given to unifying these frameworks or explicitly establishing Exponential–Gamma distributions as structural links among the Exponential, Gamma, and Lindley families. Moreover, a systematic framework that clarifies nesting relationships and transformation pathways among these models is still lacking. This gap highlights the need for further investigation into Exponential–Gamma related distributions within a coherent and integrated modelling framework. Motivated by this gap, the present study proposes a unified Exponential–Gamma framework that connects existing models and facilitates the development of new lifetime distributions.

The primary objective of this study is to develop a unified framework for Exponential–Gamma related lifetime distributions that integrates classical, mixture-based, and transformation-driven models. In addition, the study aims to demonstrate the nesting relationships among classical and higher-order models, including the Exponential, Gamma, Lindley, New Exponential–Gamma, Exponentiated New Exponential–Gamma, and Exponentiated Generalized New Exponential–Gamma distributions. A further objective is to illustrate the practical usefulness of the proposed framework through numerical evaluations, graphical illustrations, and applications to real-life lifetime data.

The main contributions of this study include providing a coherent theoretical foundation that connects existing Exponential–Gamma related distributions and clarifies their structural relationships, as well as offering a flexible platform for generating new lifetime models with enhanced tail behavior, hazard rate adaptability, and goodness-of-fit. In addition, the study presents closed-form distributional properties, numerical tables, and graphical illustrations to facilitate systematic comparison of existing and newly derived models.

The remainder of the paper is organized as follows. Section 2 reviews relevant literature on Gamma generalizations and Exponential–Gamma constructions. Section 3 presents the methodology, including the mixture and transformation mechanisms underlying the unified framework. Section 4 provides numerical illustrations and practical applications, and Section 5 concludes with a discussion of the framework's implications and potential extensions.

MATERIALS AND METHODS

This section presents the preliminaries, notation, and special cases of the probability distributions employed in this study. It further introduces a general framework for Exponential–Gamma related distributions based on mixture and transformation mechanisms. The methodological emphasis is on demonstrating how different construction strategies generate nested lifetime models within a unified structure.

Preliminaries and Notations

Let X be a positive random variable. The Gamma distribution with shape parameter $\alpha > 0$ and rate parameter $\theta > 0$, denoted by $G(\alpha, \theta)$, has the probability density function (pdf):

$$g(x;\ \alpha, \theta) = \frac{\theta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\theta x}\ ; x > 0, \tag{1}$$

and cumulative distribution function (cdf):

$$F(x;\ \alpha, \theta) = \frac{\gamma(x,\theta,\alpha)}{\Gamma(\alpha)}\ ; x > 0, \tag{2}$$

where $\gamma(\cdot,\cdot)$ denotes the upper incomplete gamma function.

When $\alpha = 1$, the Gamma distribution reduces to Exponential distribution with pdf and cdf given respectively by:

$$f(x;\ \theta) = \theta e^{-\theta x}\ ; x > 0, \tag{3}$$

$$F(x;\ \theta) = 1 - e^{-\theta x}\ ; x > 0, \tag{4}$$

Finite Mixture Construction: New Exponential-Gamma Distribution

A finite mixture of the Exponential and Gamma distributions with mixing weight π yields the New Exponential-Gamma (NEG) distribution (Umar & Yahya, 2021) defined by:

$$h(x; \alpha, \theta) = \pi f(x; \theta) + (1 - \pi) f(x;\ \alpha, \theta) \tag{5}$$

where the mixing parameter is given by

$$\pi = \frac{\theta}{\theta + \Gamma(\alpha)} \tag{6}$$

It is easily verified that when $\alpha = 1$, the NEG distribution reduces to the classical Exponential distribution. When $\alpha = 2$, the mixture in (5) yields the Lindley distribution (Lindley, 1958) with pdf and cdf respectively as:

$$h(x; 2, \theta) = \frac{\theta^2}{\theta+1}(1 + x)e^{-\theta x}; x > 0, \tag{7}$$

$$H(x; 2, \theta) = 1 - \left[1 + \frac{\theta x}{\theta+1}\right] e^{-\theta x}; x > 0, \tag{8}$$

Transformation Operator: Exponentiation

Let $F(x)$ denote the cdf of a baseline distribution. The Exponentiated form is defined as (Gupta, *et al.*, 1998 and Rather & Subramanian, 2020; Zakariyau *et al.*, 2025, among others)

$$F_{\lambda}(x) = [F(x)]^{\lambda} \tag{9}$$

This transformation introduces additional shape flexibility and has been widely used to generate extended lifetime distributions. In particular, it allows independent control of tail thickness and hazard rate curvature. For example, the Exponentiated Exponential distribution (Gupta & Kundu, 2001) has the pdf:

$$f(x; \theta, \lambda) = \lambda\theta\left(1 - e^{-\theta x}\right)^{\lambda-1} e^{-\theta x}, \tag{10}$$

while the Exponentiated Lindley distribution (Nadarajah *et al.*, 2011) is defined by:

$$f(x; \theta, \lambda) = \frac{\lambda\theta^2}{\theta+1}(1 + x)e^{-\theta x}\left[1 - \left(1 + \frac{\theta x}{\theta+1}\right)e^{-\theta x}\right]^{\lambda-1}, \tag{11}$$

Exponentiated Generalized Class of probability distributions

Let $G(x)$ denote the cdf of a baseline distribution. The Exponentiated Generalized class of distribution (see for example, Telee *et al.*, 2022 and Suleman *et al.*, 2025) is defined as:

$$F(x) = \left[1 - \{1 - G(x)\}^{\beta}\right]^{\lambda} \tag{12}$$

This class unifies several transformation-based extensions and provides additional control over skewness, tail behavior, and hazard rate shapes. Both β and λ act as shape parameters that regulate the distributional form independently.

General Exponential–Gamma Framework

Substituting the pdfs in (3) and (1) into the mixture in (5), the pdf of the New Exponential-Gamma distribution can be expressed as:

$$h(x;\ \alpha, \theta) = \frac{\theta}{\theta+\Gamma(\alpha)}\left(\theta + \theta^{\alpha-1}x^{\alpha-1}\right)e^{-\theta x}; x > 0, \tag{13}$$

and the corresponding cdf is given by:

$$H(x;\ \alpha, \theta) = \frac{1}{\theta+\Gamma(\alpha)}\left[\theta\left(1 - e^{-\theta x}\right) + \theta^{\alpha}\int_0^x t^{\alpha-1}e^{-\theta t}dt\right] \tag{14}$$

This formulation highlights that the NEG distribution arises naturally from a finite mixture of Exponential and Gamma components, with the Exponential and Lindley distributions appearing as special cases when $\alpha = 1$ and $\alpha = 2$ respectively. Hence, the NEG model serves as a central link between classical and extended lifetime distributions.

Product-Type Construction: The Exponential–Gamma Distribution

Ogunwale *et al.* (2019) introduced the Exponential-Gamma (EG) distribution through a product-type construction, with pdf:

$$f(x; \alpha, \theta) = \frac{\theta^{\alpha+1}}{\Gamma(\alpha)} x^{\alpha-1} e^{-2\theta x};\ x > 0, \tag{15}$$

Unlike the NEG model, this distribution is obtained through the product of Exponential and Gamma components (Suleman *et al.*, 2025), yet it remains structurally connected to the same Exponential–Gamma generating mechanism. This highlights the versatility of Exponential–Gamma compounding under different construction strategies.

Exponentiated New Exponential-Gamma Distribution

Applying the exponentiation operator in (10) to the cdf in (14) yields the Exponentiated New Exponential-Gamma (ENEG) distribution (Zakariyau *et al.*, 2025):

$$H_{\lambda}(x;\ \alpha,\theta) = \left[\frac{\theta}{\theta+\Gamma(\alpha)}\left(\left(1-e^{-\theta x}\right)+\theta^{\alpha-1}L\right)\right]^{\lambda} \quad (16)$$

$$h_{\lambda}(x;\alpha,\theta) = \frac{\lambda\theta^{\lambda+1}}{\left(\theta+\Gamma(\alpha)\right)^{\lambda}}\left[e^{-\theta x}+\theta^{\alpha-2}L'\right]\left[\left(1-e^{-\theta x}\right)+\theta^{\alpha-1}L\right]^{\lambda-1}, x>0 \quad (17)$$

where $L = \int_0^x t^{\alpha-1}e^{-\theta t}dt$ *and*

$L' = \frac{d}{dx}\left(\int_0^x t^{\alpha-1}e^{-\theta t}dt\right)$.

Special cases include:

1. Exponential distribution ($\lambda = \alpha = 1$),
2. Exponentiated Exponential distribution ($\alpha = 1$),
3. Lindley distribution ($\lambda = 1, \alpha = 2$),
4. Exponentiated Lindley distribution ($\alpha = 2$),
5. New Exponential-Gamma distribution ($\lambda = 1$).

Generalized and Exponentiated New Exponential–Gamma Distributions

To further enhance flexibility, the New Exponential–Gamma (NEG) distribution may first be generalized by introducing an additional shape parameter through modification of its baseline structure. Let $H(x;\ \alpha,\theta)$ denote the cdf of the NEG distribution. A generalized form, referred to as the Exponentiated Generalized New Exponential–Gamma (EGnEG) distribution (Suleman *et al.*, 2025), is obtained by incorporating an additional parameter that controls tail behavior and hazard rate shape.

Substituting the cdf of the New Exponential-Gamma distribution in the expression in (12), yields the density function, Exponentiated Generalized new Exponential Gamma (EGnEG) distribution as:

$$F(x) = \left[1-\left(\frac{\left[\theta e^{-\theta x}+\Gamma(\alpha)-\theta^{\alpha}L\right]}{\theta+\Gamma(\alpha)}\right)^{\beta}\right]^{\lambda} \quad (18)$$

where $L = \int_0^x t^{\alpha-1}e^{-\theta t}dt$

$$f(x) = \frac{\beta\theta\lambda\left(\theta+\theta^{\alpha-1}x^{\alpha-1}\right)e^{-\theta x}}{\theta+\Gamma(\alpha)}\left(\frac{\left[\theta e^{-\theta x}+\Gamma(\alpha)-\theta^{\alpha}L\right]}{\theta+\Gamma(\alpha)}\right)^{\beta-1}\left[1-\left(\frac{\left[\theta e^{-\theta x}+\Gamma(\alpha)-\theta^{\alpha}L\right]}{\theta+\Gamma(\alpha)}\right)^{\beta}\right]^{\lambda-1} \quad (19)$$

where, $x>0, \alpha, \beta, \lambda,$ and $\theta>0$. $\alpha, \beta, and\ \lambda$ Shape parameters. θ is a rate parameter and

$$L = \int_0^x t^{\alpha-1}e^{-\theta t}dt \quad (20)$$

Special cases include:

1. Exponentiated Generalized Exponential distribution ($\alpha = 1$);

$$F(x) = \left[1-\left\{1-\frac{\left[\theta\left(1-e^{-\theta x}\right)+\theta L\right]}{\theta+1}\right\}^{\beta}\right]^{\lambda} = \left[1-\left\{\frac{\left[\theta e^{-\theta x}+1-\theta L\right]}{\theta+1}\right\}^{\beta}\right]^{\lambda} \quad (21)$$

$$f(x) = \beta\theta\lambda e^{-\theta x}\left\{\frac{\left[\theta e^{-\theta x}+1-\theta L\right]}{\theta+1}\right\}^{\beta-1}\left[1-\left\{\frac{\left[\theta e^{-\theta x}+1-\theta L\right]}{\theta+1}\right\}^{\beta}\right]^{\lambda-1} \quad (22)$$

2. Exponentiated Generalized Lindley distribution ($\alpha = 2$)

$$F(x) = \left[1-\left\{1-\frac{\left[\theta\left(1-e^{-\theta x}\right)+\theta^{2}L\right]}{\theta+1}\right\}^{\beta}\right]^{\lambda} = \left[1-\left\{\frac{\left[\theta e^{-\theta x}+1-\theta^{2}L\right]}{\theta+1}\right\}^{\beta}\right]^{\lambda} \quad (23)$$

$$f(x) = \frac{\beta\theta^{2}\lambda(1+x)e^{-\theta x}}{\theta+1}\left\{\frac{\left[\theta e^{-\theta x}+1-\theta^{2}L\right]}{\theta+1}\right\}^{\beta-1}\left[1-\left\{\frac{\left[\theta e^{-\theta x}+1-\theta^{2}L\right]}{\theta+1}\right\}^{\beta}\right]^{\lambda-1} \quad (24)$$

3. Exponential distribution ($\alpha = \lambda = \beta = 1$),
4. Lindley distribution ($\alpha = 2,\ \lambda = \beta = 1$),
5. New Exponential-Gamma distribution ($\lambda = \beta = 1$),
6. Exponentiated Exponential distribution ($\alpha = \beta = 1$),
7. Exponentiated Lindley distribution ($\alpha = 2, \beta = 1$),
8. Exponentiated New Exponential-Gamma distribution ($\beta = 1$),

This construction shows that the EGnEG distribution arises naturally within the proposed framework through successive application of generalization and transformation operators. Consequently, the EGnEG distribution represents a higher-order member of the

Exponential–Gamma family and fully encompasses several well-known lifetime models as special cases.

Thus, the proposed framework accommodates Exponential–Gamma related distributions generated via finite mixtures, continuous mixing, generalization mechanisms, and transformation operators, including higher-order models such as the Exponentiated Generalized New Exponential–Gamma distribution.

RESULTS AND DISCUSSION

Parameter Estimation and Model Evaluation

The parameters of the proposed Exponential–Gamma related distributions were estimated using the Maximum Likelihood Method (MLE), which provides consistent and efficient estimates for the shape and rate parameters of the distributions. This approach allows reliable inference regarding the distributional properties and facilitates comparison across different model forms. Inference on model adequacy was performed using likelihood-based evaluation and numerical assessment of the cumulative distribution function, ensuring that key distributional features, such as nesting relationships and tail behavior, were accurately captured.

Model comparison was conducted using standard goodness-of-fit criteria, including −2 log-likelihood, Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC). These criteria enabled systematic assessment of the proposed higher-order distributions relative to classical and related competing models, demonstrating improvements in flexibility, hazard rate adaptability, and overall fit. The evaluation provides empirical support for the practical usefulness of the proposed framework and confirms the theoretical relationships among the Exponential–Gamma family of distributions.

Figure 1 presents a four-panel graphical illustration of the proposed unifying framework. Panels (a) and (c) display the probability density and distribution functions of baseline and special-case models, while Panels (b) and (d) demonstrate the effect of exponentiation and generalized exponentiation on the New Exponential–Gamma distribution.

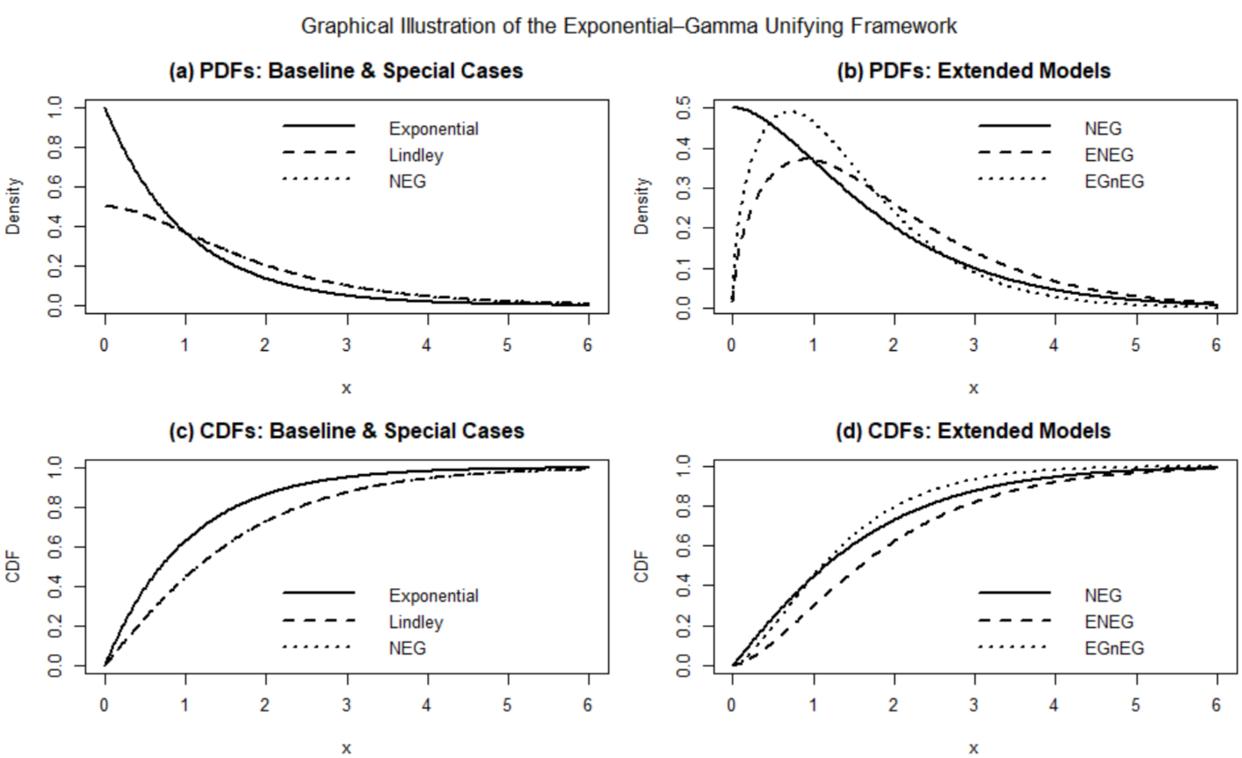

Figure 1: Graphical illustration of the unifying framework

Numerical Illustration of some special cases within the Unifying Framework

To illustrate the unifying framework, consider the New Exponential–Gamma (NEG) distribution with parameters $\alpha = 2$ and $\theta = 1$. For these values, the NEG distribution reduces to the Lindley distribution, thereby empirically validating the theoretical nesting property established in the methodological framework.

Applying the exponentiation operator with $\lambda = 1.5$ yields the Exponentiated New Exponential–Gamma (ENEG) distribution. Further introducing the generalized exponentiation parameter $\beta = 1.5$ results in the Exponentiated Generalized New Exponential–Gamma (EGnEG) distribution, demonstrating the effect of successive transformation operators on distributional flexibility.

Table 1 provides exact cumulative distribution values for selected members of the proposed Exponential–Gamma framework. The table clearly illustrates how classical distributions such as the Exponential and Gamma distributions serve as baseline models, while the Exponential, Lindley and New Exponential-Gamma distributions arise as special cases through mixture mechanisms. In particular, the numerical equivalence between the Lindley and NEG distributions for α=2 is evident across all evaluated values of x. Further flexibility is achieved through exponentiation and generalized exponentiation, leading to the ENEG and EGnEG distributions respectively. Notably, for $\alpha = 2$, the NEG distribution coincides numerically with the Lindley distribution, confirming the nesting structure of the framework. The EGnEG distribution exhibits faster accumulation of probability mass, reflecting increased flexibility and heavier tail behaviour compared with its sub-models.

Table 1: Exact Cumulative Distribution Values for Some Selected Members of the Unifying Framework

(x)	Exponential	Gamma	Lindley	NEG	ENEG	EGnEG
0.2	0.1813	0.0175	0.0994	0.0994	0.0313	0.0554
0.4	0.3297	0.0616	0.1956	0.1956	0.0865	0.1470
0.6	0.4512	0.1219	0.2865	0.2865	0.1534	0.2505
0.8	0.5507	0.1912	0.3709	0.3709	0.2259	0.3547
1.0	0.6321	0.2642	0.4482	0.4482	0.3000	0.4533
1.2	0.6988	0.3374	0.5181	0.5181	0.3729	0.5429
1.4	0.7534	0.4082	0.5808	0.5808	0.4426	0.6219
1.6	0.7981	0.4751	0.6366	0.6366	0.5079	0.6901
1.8	0.8347	0.5372	0.6859	0.6859	0.5681	0.7480
2.0	0.8647	0.5940	0.7293	0.7293	0.6229	0.7964

The inclusion of the Gamma distribution in Table 1 emphasizes its foundational role in the framework, serving as a bridge between the memoryless Exponential model and the more flexible Exponential–Gamma constructions. This numerical comparison reinforces the conceptual positioning of the Gamma distribution within the proposed hierarchy.

To further demonstrate the proposed unifying framework, a numerical illustration is presented using discrete values of x in figure 1. Histogram-style bars represent numerical evaluations corresponding to Table 1, while superimposed curves denote the theoretical cumulative distribution functions. Classical distributions appear as special cases, with successive extensions exhibiting increased flexibility and tail adaptability. The graphical results visually corroborate the numerical trends observed in Table 1 and highlight the progressive impact of mixture and transformation mechanisms.

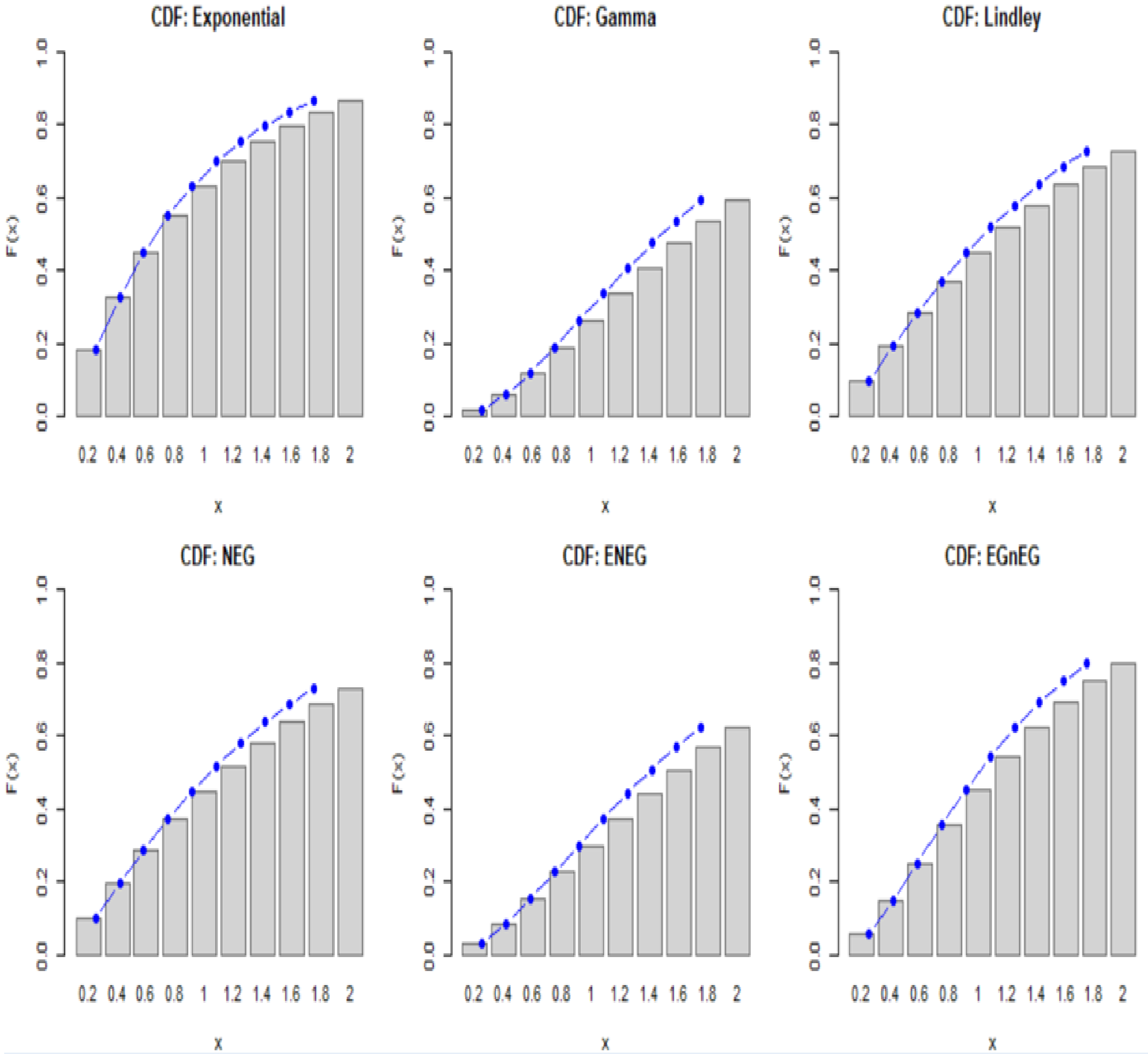

Figure 2: Cumulative distribution plots for selected members of the Exponential–Gamma unifying framework

It has been shown that fundamental distributions such as the Exponential, Gamma, and Lindley distributions arise naturally as special cases of the framework, while more flexible models—including the New Exponential–Gamma, Exponentiated New Exponential–Gamma, and Exponentiated Generalized New Exponential–Gamma distributions—emerge through successive application of exponentiation and generalization operators. Numerical illustrations and graphical representations further confirm the nesting structure of the framework and highlight the increasing adaptability of the extended models, particularly in terms of tail behavior and cumulative probability accumulation. These findings emphasize the practical relevance of the developed framework for modeling diverse hazard rate patterns.

CONCLUSION

This paper has presented a unified framework for Exponential–Gamma related lifetime distributions, providing a structured perspective that connects classical models, mixture-based constructions, and transformation-driven extensions within a single theoretical setting. By combining the Exponential and Gamma distributions through finite mixture, product-type, and transformation mechanisms, the proposed framework clarifies the relationships among several

existing lifetime models that have previously been studied in isolation. This integration resolves fragmentation in the existing literature and offers a clear hierarchical view of Exponential–Gamma constructions
The obtained unifying framework not only consolidates existing Exponential–Gamma related distributions but also provides a systematic pathway for generating new lifetime models with desirable statistical properties. Consequently, this work contributes both theoretically and practically by offering a coherent foundation for future developments in lifetime and reliability modeling. Although the present study is primarily theoretical, its implications extend to applied settings where flexible lifetime models are required. Potential extensions of this framework may include Bayesian inference, regression-based formulations, multivariate generalizations, and applications to real-life datasets arising in engineering, health sciences, and actuarial studies.

REFERENCE

Amiru, F. M., Usman, U., Shamsuddeen, S., Adamu, U. M., Abba, B. (2025). The Additive Dhillon-Chen Distribution: Properties and Applications to Failure Time Data. *International Journal of Statistical Distributions and Applications*, ***11***(1), 1-10.

Aderoju, S. A., Salau, G. M., Sanni, B. I., Adeshola, A. D., Jimoh, A. K., Wahab, T. O., & Kalu, U. (2025). The New Extended Exponential-Gamma (NEEG) Distribution: Properties and Applications to Infectious Disease Modelling. [Journal Name], [Volume(Issue)], [Page Numbers].

Gaddafi, A. B., Abbas, U. F., Manjak, N. H., Idi, H., Hamza, L., Sanni, A. Y. & Umar, M. A. (2025). A Study on Size Biased Exponential Gamma Distribution with Applications. *International Journal of Innovative Mathematics, Statistics & Energy Policies* ***13***(3); 220-243. https://doi.org/10.5281/zenodo.17153541

Gupta, R. C., Gupta, P. L., and Gupta, R. D., (1998). Modeling failure time data by Lehman alternatives. *Communications in Statistics, Theory and Methods, 27*, 887–904.

Gupta, R.D. and Kundu, D. (2001). Exponentiated exponential family: an alternative to gamma and Weibull, *Biometrical Journal*, 43(1), pp. 117-130.

Lawless, J. F. (2011). *Statistical models and methods for lifetime data*. John Wiley & Sons.

Lindley, D. V. (1958). Fiducial distributions and Bayes' theorem. *Journal of the Royal Statistical Society, Series B*, ***20***. Pp. 102–107.

Mead, M., Nassar, M. & Dey, S. (2017). A generalization of generalized gamma distributions. *Journal of Statistical Theory and Applications*, ***16***(2), 187–205.

Nadarajah, S., Bakouch, H.S., Tahmasbi, R. (2011). A generalized Lindley distribution. *Sankhya B*, ***73***, 331–359.

Ogunwale, O. D., Adewusi, O. A. & Ayeni T. M. (2019). Exponential-Gamma Distribution. *International Journal of Emerging Technology and Advanced Engineering,* ***9***(10); 245-249.

Rather, A. A., & Subramanian, C. (2020). A New Exponentiated Distribution with Engineering Science Applications. *Journal of Statistics, Applications and Probability*, ***9***(1), 127-137.

Stacy, E. W. (1962). A generalization of the gamma distribution. *The Annals of mathematical statistics*, 1187-1192.

Suleman, I., Zakariyau, N. R., Oyegoke, O. A., Yahya, W. B., Amiru, F. M. & Umar, M. A. (2025). Notes on a Modified Exponential-Gamma Distribution: Its Properties and Applications. *Journal of Basics and Applied Sciences Research (JOBASR)*, ***3***(2), 12-26. https://dx.doi.org/10.4314/jobasr.v3i2.2

Telee, L. B. S., Karki, M., & Kumar, V. (2022). Exponentiated Generalized Exponential Geometric Distribution: Model, Properties and Applications. *Interdisciplinary Journal of Management and Social Sciences,* ***3***(2), 37-60. https://doi.org/10.3126/ijmss.v3i2.50261

Umar, M. A. & Yahya, W. B. (2021): A New Exponential-Gamma Distribution with Applications. *Journal of Modern Applied Statistical Methods*.

Umar, M. A., Jimoh, J. O., Garba, M. K., Afolayon, R. B. & Yahya, W. B. (2019a). Application of Some Exponential Related Distributions. *PSSN Edited*

Proceedings of 3rd International Conference. ***3***; 482 – 487.

Umar, M. A., Jimoh, K. and Yahya, W. B. (2019b). A Note on the Applications of Some Zero-Truncated Distributions. *Professional Statisticians Society of Nigeria Edited Proceedings of 3rd International Conference*, ***3***; 499 – 505.

Yahya, W. B. & Umar, M. A. (2024): A New Poisson Exponential-Gamma Distribution for Modelling Count Data with Applications. *Quality and Quantity*. ***58***(6); 5329-5349.

Zakariyau, N. R., Umar, M. A. & Suleman, I. (2025). An Exponentiated New Exponential Gamma Distribution with Applications. *African Journal of Mathematics, Statistics and Actuarial Science (AJMSAS)*, ***1***(1); 59-81 DOI: https://doi.org/10.31920/2978- 3534/2025/v1n1a4